

BOUNDARY CALCULUS FOR CONFORMALLY COMPACT MANIFOLDS

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ABSTRACT. On conformally compact manifolds of arbitrary signature, we use conformal geometry to identify a natural (and very general) class of canonical boundary problems. It turns out that these encompass and extend aspects of already known holographic bulk-boundary problems, the conformal scattering description of boundary conformal invariants, and corresponding questions surrounding a range of physical bulk wave equations. These problems are then simultaneously solved asymptotically to all orders by a single universal calculus of operators that yields what may be described as a solution generating algebra. The operators involved are canonically determined by the bulk (*i.e.* interior) conformal structure along with a field which captures the singular scale of the boundary; in particular the calculus is canonical to the structure and involves no coordinate choices. The generic solutions are also produced without recourse to coordinate or other choices, and in all cases we obtain explicit universal formulæ for the solutions that apply in all signatures and to a range of fields. A specialisation of this calculus yields holographic formulæ for GJMS operators and Branson's Q-curvature.

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1. INTRODUCTION

Let M be a d -dimensional, compact manifold with boundary $\Sigma = \partial M$ (all structures assumed smooth). A pseudo-Riemannian metric g^o on the interior M^+ of M is said to be conformally compact if it extends to Σ by $g = r^2 g^o$ where g is non-degenerate up to the boundary, and r is a defining function for the boundary (*i.e.* Σ is the zero locus of r , and dr is non-vanishing along Σ). Assuming the conormal to Σ is not null, the restriction of g to $T\Sigma$ in $TM|_\Sigma$ determines a conformal structure, and this is independent of the choice of defining function r ; then Σ with this conformal structure is termed the conformal infinity of M^+ .

Conformally compact manifolds provide a framework for relating conformal geometry, and associated field theories, to the far field phenomena of the interior (pseudo-)Riemannian geometry of one higher dimension; the latter often termed the *bulk*. This idea seems to have had its origins in the work of Newman and Penrose, for treating four dimensional spacetime physics, and was further developed by LeBrun and others [47]. The seminal work of Fefferman–Graham [20] developed a variant of this idea, termed a Poincaré(–Einstein) metric, to develop a new approach to conformal invariant theory. On the other hand key aspects of the spectral theory for Riemannian conformally compact manifolds were pioneered by Mazzeo and Mazzeo-Melrose [50, 51, 52]. Related to this is the treatment of infinite volume hyperbolic manifolds by Patterson, Perry and others, see [56] and references therein.

More recently there has been intense interest in these structures motivated first by links to the conjectural AdS/CFT correspondence of Maldacena and second by the realisation that they are prototypes for holographic and renormalization ideas [48, 1, 43, 19, 54]. On the side of mathematics, critical progress was made in [40] based around a scalar Laplacian eigen-equation (given in our conventions)

$$(1.1) \quad (\Delta_{g^o} + s(n - s))u = 0.$$

There earlier spectral results were further developed, along with other tools, and used to relate the corresponding scattering matrix on conformally compact asymptotically Einstein manifolds to invariant objects on their boundaries at infinity; the latter including the conformally invariant powers of the Laplacian (the so-called “GJMS” operators of [36]) and Branson’s Q-curvature of [4]. A sample of the subsequent and related progress includes [42, 44, 45, 53, 61]. In all of these works the formal asymptotics play a critical role. Indeed in [40] it is emphasised at an early point that although the scattering problem they treat is certainly a global object the main new results there, and also in [21], concern “formal Taylor series statements at the boundary”. The formal calculations are based around special coordinates adapted to the structure following [38, 35].

The purpose of the current article is to take full advantage of the interior conformal structure in order to develop a new universal, and coordinate independent, approach to a class of natural problems associated canonically with conformally compact manifolds.

The focus here is on formal and asymptotic aspects along the boundary, and the problems themselves are derived directly from the geometric data of the structure. The same geometric inputs yield a calculus to treat these; this applies in all signatures and along any hypersurface that is a conformal infinity in the sense described above. This includes, for example, the cases of future and past infinities of de Sitter-like spaces. We obtain complete formulæ for the asymptotics to all orders. A specialisation of the calculus is applied to give simple holographic formulæ for the GJMS operators and the Q-curvature. We show that the problems considered include those looked at in [40], for example, but simultaneously include their wave analogues. Importantly, here these problems are based around a completely different conceptual schema, which uses conformal tractor calculus as in [3, 11, 27]; in particular this is used to show that the problems themselves arise canonically from the fundamental conformal geometry of the interior/bulk structure.

Our approach applies to weighted tractor fields of arbitrary type. This provides a universal framework for treating tensor and spinor fields of any tensor type and any conformal weight. This is realized by a comprehensive treatment of Laplace/wave equations for differential forms on Poincaré–Einstein structures in [29]. In addition, in this Article, we treat fields that we term log densities (see Section 2.1); this connects with problems treated in [21, 23].

A conformally compact manifold may be viewed as a conformal manifold (M, c) with boundary (so c is an equivalence class of conformally related metrics), that is also equipped with a generalised scale σ . This scale determines the metric g^σ on the interior and also defines the boundary; σ is a conformal density (see Section 2.1) with the boundary as its zero locus, and relative to any metric from the conformal class the function representing σ is a defining function for the boundary. Thus a conformally compact structure is precisely the data (M, c, σ) . On the other hand there is a canonical and fundamental conformally invariant, second order, and vector bundle valued, differential operator D due to Thomas [3] that is determined by just the conformal structure (M, c) . This operator applied to σ (and dividing by the dimension) gives $I := \frac{1}{d}D\sigma$, which we term the *scale tractor*. It will later be evident that there are obvious representation theoretic interpretations of this as the object breaking (or rather splitting) a conformal symmetry group pointwise via the additional structure coming from the scale. The key point here is that there is an invariant contraction of I with D to yield a second order operator $I \cdot D$ that is canonical to the structure. The problems we consider in Section 5 are equations for this degenerate Laplacian operator (or rather, degenerate wave type operator in non-Riemannian signatures), with very general and (and probably all) natural boundary conditions.

Apart from the naturality of $I \cdot D$ to pseudo-Riemannian and conformally compact structures, some comments clarify why it is a priori both natural and extremely interesting to study in this context:

- First $I \cdot D$ is a degenerate Laplacian, with degeneracy precisely along $\Sigma = \partial M$. Away from Σ , and calculated in the metric g^σ , the equation $I \cdot D u = 0$ is $(\Delta_{g^\sigma} + s(n-s))u = 0$, as in 1.1 with the spectral parameter s a fixed dimensional shift of the conformal weight, see Section 2.5, and also [27]. (Precisely, $I \cdot D u = 0$ agrees with $(\Delta_{g^\sigma} + s(n-s))u = 0$ in the case of g^σ having constant scalar curvature $Sc = -d(d-1)$, in general scalar curvature enters explicitly.) However while the operator $\Delta_{g^\sigma} + s(n-s)$ is not defined along Σ where g^σ is singular, in contrast $I \cdot D$, by dint of its conformal properties, extends to the boundary; along Σ , $I \cdot D$ is a multiple of the first order conformal Robin operator of [16].

- On Einstein manifolds I is parallel for the tractor connection, and powers of $I \cdot D$ give the conformal GJMS operators and their Q-curvatures [26]. This holds in particular on the interior of Poincaré–Einstein manifolds, and this features in the asymptotics problems in this setting, enabling a smooth expansion where otherwise log terms would arise, see Remark 5.19.
- There is strong evidence that the operator $I \cdot D$ is fundamental in physics; $I \cdot D u = 0$ is one of the key equations in a system which uniformly controls massive, massless, and partially massless free field particles [33, 34, 58], and extensions include interacting terms and supersymmetry [59]. In fact, in the setting of an AdS/CFT correspondence, our results essentially amount to explicit all orders asymptotic series for “bulk to boundary propagators” for general fields on arbitrary bulk geometries

We summarise our results and plan as follows:

- In Section 3 we prove that σ and $I \cdot D$ (or the slight variant thereof, $\frac{-1}{f^2} I \cdot D$) generate a $\mathfrak{g} := \mathfrak{sl}(2)$ algebra of operators canonically determined by the structure. This is different to the $\mathfrak{sl}(2)$ algebra of GJMS [36] coming from the Fefferman–Graham ambient metric. It leads to different asymptotic expansions which include phenomena not seen from the ambient metric alone. The structure \mathfrak{g} critically underlies the boundary calculus. Sections 4 and 5 apply this calculus, see *e.g.*, Lemma 5.3 and Proposition 5.4.
- We show in Section 4 that standard identities in the universal enveloping algebra of \mathfrak{g} lead to the identification of differential operators on M that act tangentially along Σ . In particular we obtain simple new holographic formulæ for the GJMS operators and Branson’s Q-curvature.
- In Section 5.1 and Section 5.3 we consider two different extension problems for the equation $I \cdot D u = 0$, at a generic class of weights. The first is Dirichlet in nature while the second assumes the solution vanishes along Σ to a given order that is related to weight. In fact we show that these are the only possibilities for smooth (*i.e.* power series) asymptotics. We obtain complete formal solutions described by a Bessel type solution operator; these arise from algebra associated to \mathfrak{g} . While the presence of Bessel functions is often associated with separation of variables approaches (*e.g.* [41, 62]), it is important to note that here the asymptotic expansions are given in terms of series in powers of σ , which is part of the data of the structure itself, and the solution operator also has this property. No coordinates are used, the treatment is canonical, and in Proposition 5.8 we show that the solutions are independent of any choices.
- In Section 5.4 we show the remaining weights can be treated by the introduction of expansions involving log terms. These cases involve significant subtleties. In particular to match the $I \cdot D$ equation with (1.1) it is necessary to choose a background (second) scale. After some detailed treatment the influence of this choice is made completely clear, and the entire solution (family) is controlled by an explicit solution operator. The second scale, unlike the generalised scale σ , extends to the boundary. We show that in fact only the boundary dependence of the second scale influences the solution. This result dovetails nicely with the AdS/CFT relationship between boundary renormalization group flows and bulk geometries [41, 62, 17].

- In Section 5.6 we show that when the solutions are expressed in the internal scale we get results of the usual form as in, for example, the work [40] of Graham-Zworski. In fact it is easily verified that our expressions for the asymptotics agree with [40, 21], at least in the cases of overlap. It should be emphasised that throughout our treatment is valid in any signature, and whereas the equation $(\Delta_{g^o} + s(n-s))u = 0$ has mainly been studied with u a function, here we treat the $I \cdot D$ equation on sections of (weighted) tractor bundles, and on conformally compact structures of any signature. The operator $I \cdot D$ on tractor bundles determines natural equations on tensor and spinor fields, this is central to the treatments in [33, 34, 59]. This means our results can be used to deduce results for general tensor and spinor fields in the spirit of the Eastwood curved translation principle, see [27] for an early discussion of this. In physics there is currently considerable interest in understanding higher spin systems via the AdS/CFT machinery [46, 24]. For differential forms, in [2] Aubry-Guillarmou have shown that subtle global conformal invariants [6] of the conformal infinity are captured by Poincaré-Einstein interior problems with suitable boundary behaviour. In the accompanying Article [29], we give formal, all order, solutions to those problems and explicit holographic formulæ for the operators yielding those invariants. This relies on the universal boundary calculus developed here and extends the curved translation principle.

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2. CONFORMAL GEOMETRY AND TRACTOR CALCULUS

Throughout we work on a manifold M of dimension $d \geq 3$. For simplicity we assume that this is connected and orientable. If this is equipped with a metric (of some signature (p, q)) and ∇_a denotes the corresponding Levi-Civita connection then the Riemann curvature tensor R is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where X, Y , and Z are arbitrary vector fields. In an abstract index notation (*cf.* [55]) R is denoted by $R_{ab}{}^c{}_d$, and $R(X, Y)Z$ is $X^a Y^b Z^d R_{ab}{}^c{}_d$. This can be decomposed into the totally trace-free *Weyl curvature* C_{abcd} and the symmetric *Schouten tensor* P_{ab} according to

$$(2.1) \quad R_{abcd} = C_{abcd} + 2g_{c[a} P_{b]d} + 2g_{d[b} P_{a]c},$$

where $[\dots]$ indicates antisymmetrisation over the enclosed indices. Thus P_{ab} is a trace modification of the Ricci tensor $\text{Ric}_{ab} = R_{ca}{}^c{}_b$:

$$\text{Ric}_{ab} = (n-2)P_{ab} + Jg_{ab}, \quad J := P^a{}_a.$$

2.1. Conformal densities and log densities. We need some results and techniques from conformal geometry. Recall that a conformal geometry is a d -manifold equipped with an equivalence class c of metrics (of some fixed signature (p, q)) such that if $g, \hat{g} \in c$ then $\hat{g} = e^{2\Upsilon} g$ for some $\Upsilon \in C^\infty(M)$. Conformally invariant operators on functions or between (unweighted) tensor bundles are almost non-existent, thus the notion of covariance and bi-weight is often used. In fact this covariance may be replaced by true

invariance, with conceptual and calculational simplifications if we introduce conformal densities. For more details on the approach here see [10, 31].

A conformal structure on M may also be viewed as a smooth ray subbundle $\mathcal{G} \subset S^2T^*M$ whose fibre over $x \in M$ consists of conformally related metrics at the point x . The principal bundle $\pi : \mathcal{G} \rightarrow M$ has structure group \mathbb{R}_+ , and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \text{End}(\mathbb{C})$ induces a natural (oriented) line bundle on $(M, [g])$ that we term the conformal density bundle of weight $w \in \mathbb{C}$, and denote $\mathcal{E}[w]$. The typical fibre of $\mathcal{E}[w]$ is \mathbb{C} , which we view as the complexification of an oriented copy of \mathbb{R} , since for w real there is naturally a real ray subbundle $\mathcal{E}_+[w]$ of positive elements in $\mathcal{E}[w]$. Note that, by definition, a section τ of $\mathcal{E}[w]$ is equivalent to a function $\underline{\tau}$ on \mathcal{G} with the equivariance property

$$\underline{\tau}(t^2g, p) = t^w \underline{\tau}(g, p),$$

where $t \in \mathbb{R}_+$, $g \in c$, and $p \in M$. Note that g is a section of \mathcal{G} and the pullback of $\underline{\tau}$ by this is a function f on M that represents τ in the trivialisation determined by g . If $\hat{g} = e^{2\Upsilon}g$, where $\Upsilon \in C^\infty(M)$ and \hat{f} is the pullback of $\underline{\tau}$ via \hat{g} , then

$$\hat{f} = e^{w\Upsilon} f.$$

Conformal densities are often treated informally as an equivalence class of such functions.

Each metric $g \in c$ determines a canonical section $\tau \in \Gamma\mathcal{E}[1]$, *viz.* the section with the property that $\underline{\tau}(g, p) = 1$ for all $p \in M$. It follows that there is a tautological section of $S^2T^*M \otimes \mathcal{E}[2]$ that is termed the *conformal metric*, denoted \mathbf{g} with the property that any nowhere zero section $\tau \in \Gamma\mathcal{E}[1]$ determines a metric $g \in c$ via $g := \tau^{-2}\mathbf{g}$. Henceforth the conformal metric \mathbf{g} is the default object that will be used to identify TM with $T^*M[2]$ (rather than a metric from the conformal class) and to form metric traces.

Note that since each $g \in c$ determines a trivialisation of $\mathcal{E}[w]$, it also determines a corresponding connection on $\mathcal{E}[w]$. We write ∇ for this and call it the Levi-Civita connection, since a power of this agrees with the Levi-Civita connection on $\Lambda^d T^*M$ (see *e.g.* [10]).

As well as the density bundles $\mathcal{E}[w]$, the conformal structure also determines, in an obvious way, bundles $\mathcal{F}[w]$ induced from the log representations of \mathbb{R}_+ . Here also, we may take the *weight* $w \in \mathbb{C}$, and we refer to $\mathcal{F}[w]$ as a log density bundle. In this case the fibre is \mathbb{C} as the complexification of \mathbb{R} viewed as an affine space, and the total space of the bundle is $\mathcal{G} \times \mathbb{C}$ modulo the equivalence relation

$$((t^2g, p), y) \sim ((g, p), y - w \log t).$$

In particular $\mathcal{F}[0]$ is just \mathcal{E} , the trivial bundle. For a general weight w , a section λ of $\mathcal{F}[w]$ is equivalent to a function $\underline{\lambda} : \mathcal{G} \rightarrow \mathbb{C}$ with the equivariance property

$$(2.2) \quad \underline{\lambda}(t^2g, p) = \underline{\lambda}(g, p) + w \log t,$$

where, as before, $t \in \mathbb{R}_+$, $g \in c$, and $p \in M$. As a smooth structure, $\mathcal{F}[w]$ is a trivial line bundle. However its geometric content is not compatible in the usual way with linear operations. Nevertheless it is not difficult to see that we may define such operations on sections, in an obvious way, via their representative functions on \mathcal{G} . This determines a well defined notion of adding sections of $\mathcal{F}[w_1]$ and $\mathcal{F}[w_2]$, but note that for $\lambda_1 \in \Gamma\mathcal{F}[w_1]$ and $\lambda_2 \in \Gamma\mathcal{F}[w_2]$, the sum $\lambda_1 + \lambda_2$ is a section of $\mathcal{F}[w_1 + w_2]$. Similarly we see that, for $w_0 \neq 0$, pointwise multiplication by w_0 determines a bundle isomorphism $w_0 : \mathcal{F}[1] \rightarrow \mathcal{F}[w_0]$.

Note that if τ is a positive real section of $\mathcal{E}_+[w]$ and $\underline{\tau}$ the corresponding equivariant function on \mathcal{G} , then the composition $\log \circ \underline{\tau}$ has the property (2.2), and so is equivalent to a section of $\mathcal{F}[w]$ that we shall denote $\log \tau$. It is easily seen that a real section of

$\mathcal{F}[1]$ is $\log \tau$ for some nowhere zero section $\tau \in \Gamma\mathcal{E}[1]$. On the space of real sections in $\Gamma\mathcal{F}[1]$ the Levi-Civita connection ∇ (as determined by some metric $g \in c$) determines an operator $\nabla : \Gamma\mathcal{F}[1] \rightarrow \Gamma(T^*M)$ by

$$\log \tau \mapsto \tau^{-1} \nabla \tau.$$

This is then extended to $\nabla : \Gamma\mathcal{F}[w] \rightarrow \Gamma(T^*M)$ by demanding that ∇ commute with complex multiplication, and this is consistent with also requiring ∇ act linearly over the sum of log density sections.

Finally in this Section we define the weight operator \mathbf{w} . On sections of a conformal density bundle this is just the linear operator that returns the weight. So if $\tau \in \mathcal{E}[w_0]$ then

$$\mathbf{w} \tau = w_0 \tau.$$

There is a canonical Euler operator \mathbf{v} on \mathcal{G} ; this is given by $t \frac{\partial}{\partial t}$ in the coordinates $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ on \mathcal{G} induced by coordinates x on M and a choice of $g \in c$, where in these coordinates $(t^2 g, p) \mapsto (t, x(p))$. The weight operator arises from the restriction of this to suitably equivariant sections, for example $\mathbf{w} \underline{\tau} = \mathbf{v} \underline{\tau}$. Thus \mathbf{w} satisfies a Leibniz rule: for example if $\tau_1 \in \Gamma\mathcal{E}[w_1]$ and $\tau_2 \in \Gamma\mathcal{E}[w_2]$ then $\mathbf{w}(\tau_1 \tau_2) = (w_1 + w_2) \tau_1 \tau_2$. This also determines the action of \mathbf{w} on log densities: if $\lambda \in \Gamma\mathcal{F}[w_0]$ then

$$(2.3) \quad \mathbf{w} \lambda = w_0.$$

In view of the Leibniz property we may also write $[\mathbf{w}, \lambda] = w_0$ where $[\ , \]$ indicates an operator commutator bracket.

2.2. Elements of tractor calculus. The systematic construction of conformally invariant differential operators can be facilitated by *tractor calculus* [3]. This is based around a bundle and connection that is linked to, and equivalent to, the normal conformal Cartan connection of Elie Cartan [11, 9].

On a conformal d -manifold (M, c) , the (standard) tractor bundle \mathcal{T} (or \mathcal{E}^A as the abstract index notation) is a canonical rank $d + 2$ vector bundle equipped with the canonical (normal) tractor connection $\nabla^{\mathcal{T}}$. A choice of metric $g \in c$ determines an isomorphism

$$(2.4) \quad \mathcal{T} \stackrel{g}{\cong} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1].$$

In the following we shall frequently use (2.4). Sometimes this will be without any explicit comment but also we may write for example $T \stackrel{g}{=} (\sigma, \mu_a, \rho)$, or alternatively $[T]_g = (\sigma, \mu_a, \rho)$, to mean T is an invariant section of \mathcal{T} and (σ, μ_a, ρ) is its image under the isomorphism (2.4). In terms of this splitting the tractor connection is given by

$$(2.5) \quad \nabla_a^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho + P_{ab} \sigma \\ \nabla_a \rho - P_{ac} \mu^c \end{pmatrix}.$$

Changing to a conformally related metric $\hat{g} = e^{2\Upsilon} g$ gives a different isomorphism, which is related to the previous by the transformation formula

$$(2.6) \quad \widehat{(\sigma, \mu_b, \rho)} = (\sigma, \mu_b + \sigma \Upsilon_b, \rho - g^{cd} \Upsilon_c \mu_d - \frac{1}{2} \sigma g^{cd} \Upsilon_c \Upsilon_d),$$

where Υ_a is the one-form $d\Upsilon$. It is straightforward to verify that the right-hand-side of (2.5) also transforms in this way and this verifies the conformal invariance of $\nabla^{\mathcal{T}}$. In the following we will usually write simply ∇ for the tractor connection. Since it is the

only connection we shall use on \mathcal{T} , its dual, and tensor powers, this should not cause any confusion.

There is also a conformally invariant *tractor metric* h on \mathcal{T} . This is given (as a quadratic form on T^A as above) by

$$(2.7) \quad (\sigma, \mu, \rho) \mapsto \mathbf{g}^{-1}(\mu, \mu) + 2\sigma\rho =: h(T, T) = h_{AB}T^AT^B ;$$

it is preserved by the connection. We shall often write T^2 as a shorthand for the right hand side of this display. Note that this has signature $(p+1, q+1)$ on a conformal manifold (M, c) of signature (p, q) . The tractor metric h_{AB} and its inverse h^{AB} are used to identify \mathcal{T} with its dual in the obvious way.

Tensor powers of the standard tractor bundle \mathcal{T} , and tensor parts thereof, are vector bundles that are also termed tractor bundles. We shall denote an arbitrary tractor bundle by \mathcal{E}^Φ and write $\mathcal{E}^\Phi[w]$ to mean $\mathcal{E}^\Phi \otimes \mathcal{E}[w]$; w is then said to be the weight of $\mathcal{E}^\Phi[w]$. In the obvious way, the operator \mathbf{w} of Section 2.1 is extended to sections of weighted tractor bundles $\mathcal{E}^\Phi[w] \ni f$ by

$$\mathbf{w}f = wf.$$

Whereas the tractor connection maps sections of a weight 0 tractor bundle \mathcal{E}^Φ to sections of $T^*M \otimes \mathcal{E}^\Phi$, there is a conformally invariant operator which maps between sections of weighted tractor bundles. This is the Thomas-D (or tractor-D) operator

$$D_A : \Gamma\mathcal{E}^\Phi[w] \mapsto \Gamma(\mathcal{E}^A \otimes \mathcal{E}^\Phi[w-1]),$$

given in a scale g by

$$(2.8) \quad [D_AV]_g = \begin{pmatrix} (d+2w-2)\mathbf{w}V \\ (d+2\mathbf{w}-2)\nabla_a V \\ -(\Delta + \mathbf{J}\mathbf{w})V \end{pmatrix},$$

where $\Delta = \mathbf{g}^{ab}\nabla_a\nabla_b$, and ∇ is the coupled Levi-Civita-tractor connection [3, 60].

A key point to emphasise here is that the Thomas-D operator is a fundamental object in conformal geometry. On a conformal manifold the tractor bundle is “as natural” as the tangent bundle. On the other hand the tractor-D operator on densities on $\Gamma\mathcal{E}[1]$ basically defines the tractor bundle, see [11].

2.3. Almost Einstein structures and Poincaré–Einstein manifolds. We recall here some facts from the literature that we will need later. These partly motivate our overall approach. The first is that conformal geometry has a strong bias toward Einstein metrics [57]. Following [30] we state this as follows.

Theorem 2.1. *On a conformal manifold (M^d, c) (of any signature) there is a 1-1 correspondence between conformal scales $\sigma \in \Gamma\mathcal{E}[1]$, such that $g^\sigma = \sigma^{-2}\mathbf{g}$ is Einstein, and parallel standard tractors I with the property that X_AI^A is nowhere vanishing. The mapping from Einstein scales to parallel tractors is given by $\sigma \mapsto \frac{1}{d}D_A\sigma$ while the inverse is $I^A \mapsto X^AI_A$.*

The statement as here is easily verified using (2.5), or may be viewed as an easy consequence of the *definition* of the tractor connection from [3]. However the normal tractor connection is canonical from other points of view, and so this suggests a deep link between conformal geometry and Einstein structures, or more generally to almost Einstein structures [27] as follows.

Definition 2.2. We say that a conformal manifold (M^d, c) , $d \geq 3$, is *almost Einstein* if it is equipped with a non-zero parallel standard tractor I .

Since this condition is equivalent to a certain holonomy reduction of the normal tractor connection, it follows at once that in general a conformal manifold (M, c) will not admit an almost Einstein structure. It is only a slight generalisation of the Einstein condition, and in fact it follows easily from the Theorem above, that on an almost Einstein manifold σ is non-zero on an open dense set [30]. Note that the zero locus $\mathcal{Z}(\sigma)$ of the “scale” σ is a conformal infinity, if non-empty.

On a Riemannian conformally compact manifold, if the defining function (as in Section 1) satisfies $|dr|_g^2 = 1$ along M , the sectional curvatures of g^o tend to -1 at infinity and the structure is said to be *asymptotically hyperbolic (AH)* [50]. The model geometry here is the Poincaré hyperbolic ball and thus the corresponding metrics are sometimes called Poincaré metrics. Generalising the hyperbolic ball in a more strict way, one may suppose that the interior conformally compact metric g^o is Einstein with the normalisation $\text{Ric}(g^o) = -ng^o$, where $n = d - 1$, and in this case the structure is said to be Poincaré-Einstein (PE); in fact PE manifolds are necessarily asymptotically hyperbolic. As noted in [25], and then discussed in more detail in [27], we have the following result.

Proposition 2.3. *A Poincaré-Einstein manifold is a (Riemannian signature) almost Einstein manifold (M, c, I) with boundary satisfying $\partial M = \mathcal{Z}(\sigma)$, and such that $I^2 = 1$.*

It follows that the notion of an almost Einstein structure provides an approach for treating Poincaré-Einstein structures that is closely related to the Cartan/tractor calculus [28].

Surprisingly, for many problems the same techniques apply to *any conformally compact structure*, and this is a critical observation we make and use here.

2.4. The scale tractor. To avoid awkward language we shall use the term (*generalised*) *scale* for any section $\sigma \in \Gamma\mathcal{E}[1]$ that is not identically zero; only in the case that σ is nowhere zero is this a *true scale*, so that $\sigma^{-2}g$ is a metric.

Definition 2.4. For σ a scale, on any conformal manifold, we call

$$I_A := \frac{1}{d} D_A \sigma$$

the corresponding *scale tractor*. Note that $\sigma = X^A I_A$.

So an almost Einstein manifold has a parallel scale tractor. Specialising further: In the model of the hemisphere equipped with its standard hyperbolic metric, I may be identified with the vector in \mathbb{R}^{d+2} the fixing of which reduces the conformal group $SO(d+1, 1)$ to a copy of $SO(d, 1)$; the latter with orbit spaces the bounding equatorial conformal infinity, and the interior [28].

One congenial feature of the scale tractor I is that its squared length gives a generalisation of the scalar curvature as follows (see [28] which deals with Riemannian signature but the result applies in any case). First note that using (2.7) and (2.8) we have

$$I^2 \stackrel{g^o}{=} -\frac{2\sigma}{d} (\Delta\sigma + \mathcal{J}\sigma) + (\nabla^a \sigma)(\nabla_a \sigma).$$

Thus if I^2 is nowhere vanishing then σ is non-vanishing on an open dense set in M . Where σ is non-zero let us write $g^o := \sigma^{-2}g$. Then, at any point q with $\sigma(q) \neq 0$,

$$I^2 \stackrel{g^o}{=} -\frac{2}{d} \sigma^2 \mathcal{J} = -\frac{2}{d} J^{g^o},$$

since σ is preserved by the metric it determines. Note that here $J = g_{ab}P^{ab}$, whereas $Jg^o = g_{ab}^o P^{ab}$ is the trace of the Schouten tensor *as calculated in the scale g^o* .

Furthermore note that if I^2 is nowhere zero then at any $p \in M$ such that $\sigma(p) = 0$, we have $\nabla_a \sigma \neq 0$; this follows since $g^{ab}(\nabla_a \sigma)(\nabla_b \sigma)$ is of the form $I^2 - 2\sigma\rho$ (for some smooth density ρ). It follows that the zero locus $\mathcal{Z}(\sigma)$ of σ is either empty or forms a smooth hypersurface (*i.e.* a smoothly embedded codimension 1 submanifold), *cf.* [28].

Let us summarise aspects of this discussion.

Lemma 2.5. *Let I be a scale tractor on any conformal manifold. Where the scale $\sigma := X^A I_A$ is non-zero we have*

$$I^2 := I^A I_A = -\frac{\text{Sc}^{g^o}}{d(d-1)},$$

with $g^o := \sigma^{-2}g$. If I^2 is nowhere zero then σ is non-zero on an open dense set, and $\mathcal{Z}(\sigma)$ (if non-empty) is a smooth hypersurface.

In this language asymptotically hyperbolic manifolds are conformally compact manifolds (M, c, σ) with the scale tractor satisfying $I^2 = 1$ along ∂M . On the other hand on any manifold with (M, c, σ) (where σ is a generalised scale), $I^2 = \text{constant}$ means that g^o has constant scalar curvature where defined, with, in particular, $I^2 = 0$ meaning that g^o has zero scalar curvature. On a given conformal manifold it is evident that seeking scales such that I^2 is constant is a natural problem generalising the Yamabe and Loewner-Nirenberg problems. Following [28] we say a conformal manifold with constant I^2 is almost scalar constant (ASC). In the case of a Riemannian signature ASC manifold with $I^2 \neq 0$, $\mathcal{Z}(\sigma)$ can only be non-empty if I^2 is positive, and if so the structure is asymptotically hyperbolic along $\mathcal{Z}(\sigma)$. On the other hand, in other signatures there are more possibilities. For example de Sitter space is a Lorentzian signature ASC structure with I^2 negative. The zero locus $\mathcal{Z}(\sigma)$ is the boundary and comprises two components which are interpreted as a past conformal infinity and a future conformal infinity; each boundary component is spacelike (as follows easily from the sign of I^2).

2.5. The canonical degenerate Laplacian. Putting the above results together we note that there is a completely canonical differential operator on conformal manifolds equipped with a scale. Namely

$$I \cdot D := I^A D_A.$$

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight:

$$I \cdot D : \mathcal{E}^\Phi[w] \rightarrow \mathcal{E}^\Phi[w-1].$$

Now there is a rather subtle point, the importance of which will soon be clear, namely that, since the tractor-D operator D_A is conformally invariant, the operator $I \cdot D$ is conformally invariant except by the explicit coupling to the scale tractor.

Expanding $I \cdot D$ in terms of some background metric g , we have

$$I \cdot D \stackrel{g}{=} \begin{pmatrix} -\frac{1}{d}(\Delta\sigma + J\sigma) & \nabla^a \sigma & \sigma \end{pmatrix} \begin{pmatrix} \mathbf{w}(d+2\mathbf{w}-2) \\ \nabla_a(d+2\mathbf{w}-2) \\ -(\Delta + J\mathbf{w}) \end{pmatrix},$$

and as an operator on any density or tractor bundle of some weight, *e.g.* w so $\mathcal{E}^\Phi[w]$, each occurrence of \mathbf{w} evaluates to w . So then

$$(2.9) \quad I \cdot D \stackrel{g}{=} -\sigma \Delta + (d + 2w - 2)[(\nabla^a \sigma) \nabla_a - \frac{w}{d}(\Delta \sigma)] - \frac{2w}{d}(d + w - 1)\sigma \mathbf{J}.$$

Now if we calculate in the metric $g^o = \sigma^{-2}g$, away from the zero locus of σ , and trivialise the densities accordingly, then $\sigma = 1$ in such a scale and we have

$$I \cdot D \stackrel{g^o}{=} -\left(\Delta^{g^o} + \frac{2w(d + w - 1)}{d} \mathbf{J}^{g^o}\right).$$

In particular if g^o satisfies $\mathbf{J}^{g^o} = -\frac{d}{2}$ (*i.e.* $\text{Sc}^{g^o} = -d(d - 1)$ or equivalently $I^2 = 1$) then, relabeling $d + w - 1 =: s$ and $d - 1 =: n$, we have

$$(2.10) \quad I \cdot D \stackrel{g^o}{=} -(\Delta^{g^o} + s(n - s)),$$

as commented in the Introduction.

On the other hand, looking again to (2.9), we see that along $\Sigma = \mathcal{Z}(\sigma)$ (assumed non-empty) the operator $I \cdot D$ is first order. For example (see [28]), if the manifold is ASC with $I^2 = 1$, then along Σ

$$I \cdot D = (d + 2w - 2)\delta_n,$$

on $\mathcal{E}^\Phi[w]$, where δ_n is the conformal Robin operator,

$$\delta_n \stackrel{g}{=} n^a \nabla_a^g - w H^g,$$

of [16, 5] (twisted with the tractor connection); here n^a is a unit normal and H^g the mean curvature, as measured in the metric g .

To further connect with other canonical problems studied in the literature, it is interesting to explicate the operator $I \cdot D$ on log densities. In particular let $U \in \Gamma \mathcal{F}[w]$, then

$$I \cdot D U \stackrel{g}{=} [-\sigma \Delta + (d - 2)(\nabla^a \sigma) \nabla_a] U - \frac{w}{d}[(d - 2)\Delta \sigma + 2(d - 1)\sigma \mathbf{J}].$$

Thus in the preferred scale

$$(2.11) \quad I \cdot D U \stackrel{g^o}{=} -\Delta^{g^o} U + w(d - 1).$$

This last operator at $w = -1$ is precisely the one studied in [21].

3. A BOUNDARY CALCULUS FOR THE DEGENERATE LAPLACIAN

Let (M, c) be a conformal structure of dimension $d \geq 3$ and of any signature. Given σ a section of $\mathcal{E}[1]$, write I_A for the corresponding scale tractor. That is $I_A = \frac{1}{d} D_A \sigma$. Then $\sigma = X^A I_A$.

3.1. The $\mathfrak{sl}(2)$. Suppose that $f \in \mathcal{E}^\Phi[w]$, where \mathcal{E}^Φ denotes any tractor bundle. Select $g \in c$ for the purpose of calculation, and write $I_A \stackrel{g}{=} (\sigma, \nu_a, \rho)$ to simplify the notation. Then using $\nu_a = \nabla_a \sigma$, we have

$$\begin{aligned} I \cdot D(\sigma f) &= (d + 2w)((w + 1)\sigma \rho f + \sigma \nu_a \nabla^a f + f \nu_a \nu^a) \\ &\quad - \sigma(\sigma \Delta f + 2\nu_a \nabla^a f + f \Delta \sigma + (w + 1)\mathbf{J} \sigma f), \end{aligned}$$

while

$$-\sigma I \cdot D f = -\sigma(d + 2w - 2)(w \rho f + \nu_a \nabla^a f) + \sigma^2(\Delta f + w \mathbf{J} f).$$

So, by virtue of the fact that $\rho = -\frac{1}{d}(\Delta\sigma + J\sigma)$, we have

$$[I \cdot D, \sigma]f = (d + 2w)(2\sigma\rho + \nu_a\nu^a)f.$$

Now $I^A I_A = I^2 \stackrel{g}{=} 2\sigma\rho + \nu_a\nu^a$, whence the last display simplifies to

$$[I \cdot D, \sigma]f = (d + 2w)I^2 f.$$

Denoting by \mathbf{w} the weight operator on tractors, we have the following.

Lemma 3.1. *Acting on any section of a weighted tractor bundle we have*

$$[I \cdot D, \sigma] = I^2(d + 2\mathbf{w}),$$

where \mathbf{w} is the weight operator.

Remark 3.2. A similar computation to above shows that, more generally,

$$I \cdot D(\sigma^\alpha f) - \sigma^\alpha I \cdot Df = -\sigma^{\alpha-1} \alpha I^2(d + 2\mathbf{w} + \alpha - 1)f,$$

for any constant α .

Remark 3.3. The identity of Lemma 3.1, and the algebra it generates (as below), is unchanged upon adding to the operator $I \cdot D$ an additional term of the form:

$$I \cdot D \rightarrow I \cdot D + \mathcal{W} \overbrace{\sharp \cdots \sharp}^{k \text{ times}},$$

where \mathcal{W} is any weight -1 tractor tensor in $\Gamma(\otimes^k \text{End}(\mathcal{T}))$ and \sharp denotes the natural tensorial action of endomorphisms on tractor sections (to the right), so for example on a rank one tractor T^A

$$\mathcal{W}^\sharp T^A := -\mathcal{W}^A_B T^B.$$

Hence the canonical operator $I^A \not{D}_A$ of Theorem 4.7 of [28], for example, obeys the same algebra as $I \cdot D$ with σ and the weight operator \mathbf{w} . Thus the extension problem uncovered there for the W -tractor for almost Einstein structures is also solved formally by the algebraic methods introduced in Section 5 below.

The operator $I \cdot D$ lowers conformal weight by 1. On the other hand, as an operator (by tensor product) σ raises conformal weight by 1. We can record this by the commutator relations

$$[\mathbf{w}, I \cdot D] = -I \cdot D \quad \text{and} \quad [\mathbf{w}, \sigma] = \sigma,$$

so with the Lemma we see that the operators σ , $I \cdot D$, and \mathbf{w} , acting on weighted scalar or tractor fields, generate an $\mathfrak{sl}(2)$ Lie algebra, provided I^2 is nowhere vanishing. It is convenient to fix a normalisation of the generators; we record this and our observations as follows.

Proposition 3.4. *Suppose that (M, c, σ) is such that I^2 is nowhere vanishing. Setting $x := \sigma$, $y := -\frac{1}{I^2} I \cdot D$, and $h := d + 2\mathbf{w}$ we obtain the commutation relations*

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,$$

of standard $\mathfrak{sl}(2)$ generators.

In the case of $I^2 = 0$ the result is an Inönü-Wigner contraction of the $\mathfrak{sl}(2)$:

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = 0,$$

where h and x are as before, but now $y = -I \cdot D$.

Subsequently \mathfrak{g} will be used to denote this $(\mathfrak{sl}(2))$ Lie algebra of operators. From Proposition 3.4 (and in concordance with remark 3.2) follow some useful identities in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Corollary 3.5.

$$(3.1) \quad \begin{aligned} [x^k, y] &= x^{k-1}k(d + 2\mathbf{w} + k - 1) = x^{k-1}k(h + k - 1) \\ &\text{and} \\ [x, y^k] &= y^{k-1}k(d + 2\mathbf{w} - k + 1) = y^{k-1}k(h - k + 1). \end{aligned}$$

4. TANGENTIAL OPERATORS AND HOLOGRAPHIC FORMULÆ

Suppose that $\sigma \in \Gamma\mathcal{E}[1]$ is such that $I_A = \frac{1}{d}D_A\sigma$ satisfies that $I^A I_A = I^2$ is nowhere zero. As explained in Section 2.4, the zero locus $\mathcal{Z}(\sigma)$ of σ is then either empty or forms a smooth hypersurface (*i.e.* a smoothly embedded codimension 1 submanifold), *cf.* [28] where the Riemannian signature case is discussed.

Conversely if Σ is any smooth oriented hypersurface then, at least in a neighbourhood of Σ , there is a smooth defining function f . That is, Σ is the zero locus of a smooth function s , with ds nowhere vanishing along Σ . Now take $\sigma \in \Gamma\mathcal{E}[1]$ to be the unique density which gives s in the trivialisation of $\mathcal{E}[1]$ determined by some $g \in c$. It follows then that $\Sigma = \mathcal{Z}(\sigma)$ and $\nabla^g \sigma$ is non-zero at all points of Σ . If $\nabla^g \sigma$ is nowhere null along Σ we say Σ is *nowhere null* and then I^2 is nowhere vanishing in a (possibly smaller) neighbourhood of Σ , and we are in the situation of the previous paragraph. We shall call such a σ a *defining density* for Σ , and to simplify the discussion we shall take M to agree with this neighbourhood of Σ and write $M_o := M \setminus \Sigma$. Until further notice σ will mean such a section of $\mathcal{E}[1]$ with $\Sigma = \mathcal{Z}(\sigma)$ non-empty and nowhere null. Note that Σ has a conformal structure c_Σ induced in the obvious way from (M, c) and is a conformal infinity for the metric $g^o := \sigma^{-2}g$ on $M \setminus \Sigma$.

4.1. Tangential operators. Suppose that σ is a defining density for a hypersurface Σ . Let $P : \Gamma\mathcal{E}^\Phi[w_1] \rightarrow \Gamma\mathcal{E}^\Phi[w_2]$ be some linear operator in a neighbourhood of Σ . We shall say that P *acts tangentially (along Σ)* if $P \circ \sigma = \sigma \circ \tilde{P}$, where $\tilde{P} : \Gamma\mathcal{E}^\Phi[w_1 - 1] \rightarrow \Gamma\mathcal{E}^\Phi[w_2 - 1]$ is some other linear operator on the same neighbourhood. The point is that for a tangential operator P we have

$$P(f + \sigma h) = Pf + \sigma \tilde{P}h.$$

Thus along Σ the operator P is insensitive to how f is extended off Σ . It is easily seen that, for example, in the case that P is a tangential differential operator, there is a formula for P , along Σ , involving only derivatives tangential to Σ , and the converse also holds.

Using Corollary 3.5 we see at once that certain powers of $I \cdot D$ act tangentially on appropriately weighted tractor bundles. We state this precisely. Suppose that Σ is a (nowhere null) hypersurface in a conformal manifold (M^{n+1}, c) , and σ a defining density for Σ . Then recall $\Sigma = \mathcal{Z}(\sigma)$ and I^2 is nowhere zero in a neighbourhood of Σ , where $I_A := \frac{1}{n+1}D_A\sigma$ is the scale tractor. The following holds.

Theorem 4.1. *Let \mathcal{E}^Φ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along Σ*

$$(4.1) \quad P_k : \Gamma\mathcal{E}^\Phi\left[\frac{k-n}{2}\right] \rightarrow \Gamma\mathcal{E}^\Phi\left[\frac{-k-n}{2}\right] \quad \text{given by} \quad P_k := \left(-\frac{1}{I^2}I \cdot D\right)^k$$

is a tangential differential operator, and so determines a canonical differential operator $P_k : \Gamma\mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \rightarrow \Gamma\mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$.

Proof. The P_k are differential by construction. Thus the result is immediate from Corollary 3.5 with $\tilde{P} = \left(-\frac{1}{I^2}I \cdot D\right)^k$. \square

Remark 4.2. As in the Theorem above, we shall use the same notation P_k to mean the tangential differential operator defined by (4.1), and also the operator $P_k : \Gamma\mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \rightarrow \Gamma\mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$ that this determines. The meaning will be clear by context.

Recall we write $n = d - 1$. The operator P_n is said to be of *critical order*, since its order along Σ is n , the dimension of Σ . As an operator on functions we have the following.

Proposition 4.3. *The differential operator on functions*

$$P_n : \Gamma\mathcal{E}_\Sigma \rightarrow \Gamma\mathcal{E}_\Sigma[-n]$$

annihilates constant functions.

Proof. Consider $P_n 1$. Since the ambient operator P_n is tangential there is no loss of generality in extending the constant function 1 on Σ to the constant function 1 on M and calculating $[P_n 1]_\Sigma$. But $D_A 1 = 0$, and so $P_n 1 = 0$. \square

4.2. Holographic formulæ for the GJMS operators and Q-curvature. Theorem 4.1 does not establish whether or not the P_k are non-trivial. In cases where they are non-trivial then they will be intrinsic to the hypersurface if the metric g° is determined by the conformal structure on Σ to sufficient order.

In the case where g° is Einstein we can, for the most part, fill in these details. Note that g° Einstein implies I^2 is a non-zero constant. By a constant scaling of σ we may take this to be ± 1 (and recall $\text{Sc}^{g^\circ} = -n(n+1)I^2$), so without loss of generality we assume this.

First we record the following.

Proposition 4.4. *Suppose that (M_o, g°) is Einstein, with $\text{Sc}^{g^\circ} = \mp n(n+1)$. Then for k odd $P_k : \Gamma\mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \rightarrow \Gamma\mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$ is the zero operator. For k even the operator has leading term $(-1)^k ((k-1)!!)^2$ times $\Delta_\Sigma^{\frac{k}{2}}$, (up to the sign \pm) where Δ_Σ is the intrinsic Laplacian for (Σ, g_Σ) with $g_\Sigma \in c_\Sigma$.*

The picture can be refined significantly for operators on densities. We write $\mathcal{E}_\Sigma[w]$ for the intrinsic weight w conformal density bundle of (Σ, c_Σ) , and note that this may be canonically identified with $\mathcal{E}[w]|_\Sigma$.

Theorem 4.5. *Suppose that (M_o, g°) is Einstein, with $\text{Sc}^{g^\circ} = \mp n(n+1)$. Then the operator on densities $P_k : \Gamma\mathcal{E}_\Sigma[\frac{k-n}{2}] \rightarrow \Gamma\mathcal{E}_\Sigma[\frac{-k-n}{2}]$ satisfies the following:*

- For k odd P_k is the zero operator.
- For even k , with $k \leq n$, P_k is $(-1)^k ((k-1)!!)^2$ times the order k GJMS operator \mathcal{P}_k on (Σ, c_Σ) .

The proof of this is given in Section 6.1 below.

Remark 4.6. For the ranges as discussed, (and in the spirit of e.g. [37, 49, 18]) we can view the k -even P_k as giving *holographic formulæ* for the GJMS operators. These are given as an explicit formula in terms of the ambient almost Einstein (or Poincaré–Einstein) space which has one higher dimension than the space where the operator acts.

In the case of n odd the Theorem does not recover all the operators of [36]. This is just an issue of the extent to which the boundary conformal structure determines the ambient metric g^o . For example if g^o is the formal Poincaré metric in the sense of [22, 40] then P_k will be intrinsic for all k , and it is not difficult to argue (following [40]) that for all $k \in 2\mathbb{Z}_{\geq 1}$ these are the GJMS operators.

We can now give a similar holographic formula for the Q -curvature.

Theorem 4.7. *Suppose that (M_o^{n+1}, g^o) is Einstein, with $\text{Sc}^{g^o} = \mp n(n+1)$ and n even. Let $\mu \in \Gamma\mathcal{E}_\Sigma[1]$ be a scale for (Σ, c_Σ) . This determines a metric $g_\Sigma^\mu \in c_\Sigma$ and the corresponding (Branson) Q -curvature of (Σ, g_Σ^μ) is given by*

$$(4.2) \quad Q^{g_\Sigma^\mu} = \frac{1}{((n-1)!!)^2} \left[\left(-\frac{1}{I^2} I \cdot D \right)^n \log \tilde{\mu} \right] \Big|_\Sigma,$$

where $\tilde{\mu}$ is any smooth extension of μ to a section of $\mathcal{E}[1]$ in a neighbourhood of Σ .

Proof. Suppose that $\hat{\mu}$ is a different smooth extension of μ . Then $\hat{\mu} = f\tilde{\mu}$, for some smooth function f with the property that $f|_\Sigma = 1$. Thus

$$\left[\left(-\frac{1}{I^2} I \cdot D \right)^n \log \hat{\mu} \right] \Big|_\Sigma = \left[\left(-\frac{1}{I^2} I \cdot D \right)^n \log \tilde{\mu} \right] \Big|_\Sigma + \left[\left(-\frac{1}{I^2} I \cdot D \right)^n \log f \right] \Big|_\Sigma.$$

But

$$\left[\left(-\frac{1}{I^2} I \cdot D \right)^n \log f \right] \Big|_\Sigma = [P_n f] \Big|_\Sigma = [P_n 1] \Big|_\Sigma = 0,$$

since the critical operator P_n is tangential and, by Proposition 4.3, annihilates constants. Thus we see that the right-hand-side of (4.2) is independent of how μ is extended to $\tilde{\mu}$ (that is, $y^n \circ \log$ is a tangential operator on $\mathcal{E}[1]$).

Now the proof follows easily by a Fefferman–Graham ambient metric argument analogous to the proof of Theorem 4.5, using also the result of Fefferman–Hirachi [23] that, in terms of the Fefferman–Graham ambient metric, the Q -curvature is given by $\Delta_{\tilde{\Sigma}}^{n/2} \log \mu|_{\tilde{\Sigma}}$ where μ is a homogeneous degree 1 function on \tilde{M} that along $\tilde{\Sigma}$ lifts μ . \square

Remark 4.8. Note that in any dimension, and on any conformally compact manifold the formula (4.2) defines a curvature quantity Q^{g^μ} determined by the metric g_Σ^μ and the ambient geometry. (Note that the first part of the Proof of Theorem 4.5 does not use that n is even, nor that g^o is Einstein.) By construction this satisfies an analogue of the celebrated Q -curvature conformal transformation law:

$$Q^{e^{2\omega} g^\mu} = Q^{g^\mu} + \mathcal{P}_n \omega,$$

with $(-1)^n ((n-1)!!)^2 \mathcal{P}_n = P_n$ defining \mathcal{P}_n . When n is even it is reasonable to view this as a (in general non-intrinsic) generalisation of the Q -curvature.

Remark 4.9. The Q -curvature as discussed above is sometimes called the *critical Q -curvature* since it is associated to the dimension order GJMS operators. Related scalar curvature quantities associated to the other GJMS operators are sometimes called *non-critical Q -curvatures* and are also of interest for (for example) curvature prescription problems [7]. In an obvious adaption of the ideas from Theorem 4.7, holographic formulae for these arise (at least up to the orders covered by Theorem 4.5) by applying the operators $\left(-\frac{1}{I^2} I \cdot D \right)^k$ to a suitable power of μ . We leave the details to the reader.

5. THE EXTENSION PROBLEMS AND THEIR ASYMPTOTICS

Henceforth we consider a structure (M, c, σ) with σ a defining density for a hypersurface Σ and I^2 nowhere zero. We consider the problem of solving, off Σ asymptotically,

$$I \cdot Df = 0,$$

for $f \in \Gamma \mathcal{E}^\Phi[w_0]$ and some given weight w_0 . For simplicity we henceforth calculate on the side of Σ where σ is non-negative, so effectively this amounts to working locally along the boundary of a conformally compact manifold.

5.1. Solutions of the first kind. Here we treat an obvious Dirichlet-like problem where we view $f|_\Sigma$ as the initial data. Suppose that f_0 is an arbitrary smooth extension of $f|_\Sigma$ to a section of $\mathcal{E}^\Phi[w_0]$ over M . We seek to solve the following problem:

Problem 5.1. Given $f|_\Sigma$, and an arbitrary extension f_0 of this to $\mathcal{E}^\Phi[w_0]$ over M , find $f_i \in \mathcal{E}^\Phi[w_0 - i]$ (over M), $i = 1, 2, \dots$, so that

$$f^{(\ell)} := f_0 + \sigma f_1 + \sigma^2 f_2 + \dots + O(\sigma^{\ell+1})$$

solves $I \cdot Df = O(\sigma^\ell)$, off Σ , for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

Remark 5.2. For $i \geq 1$ we do not assume that the f_i are necessarily non-vanishing along Σ .

We write $h_0 = d + 2w_0$ so that $hf_0 = h_0 f_0$, for example. The existence or not of a solution at generic weights is governed by the following result.

Lemma 5.3. *Let $f^{(\ell)}$ be a solution of Problem 5.1 to order $\ell \in \mathbb{Z}_{\geq 0}$. Then provided $\ell \neq h_0 - 2 = n + 2w_0 - 1$ there is an extension*

$$f^{(\ell+1)} = f^{(\ell)} + \sigma^{\ell+1} f_{\ell+1},$$

unique modulo $\sigma^{\ell+2}$, which solves

$$I \cdot Df^{(\ell+1)} = 0 \quad \text{modulo} \quad O(\sigma^{\ell+1}).$$

If $\ell = h_0 - 2$ then the extension is obstructed by $P_{\ell+1} f_0|_\Sigma$.

Proof. Note that $I \cdot Df = 0$ is equivalent to $-\frac{1}{I^2} I \cdot Df = 0$ and so we can recast this as a formal problem using the Lie algebra $\langle x, y, h \rangle$ from Proposition 3.4. Using the notation from there

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell+1)(h+\ell)f_{\ell+1} + O(x^{\ell+1}).$$

Now $hf_{\ell+1} = (h_0 - 2(\ell+1))f_{\ell+1}$, thus

$$(5.1) \quad yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell+1)(h_0 - \ell - 2)f_{\ell+1} + O(x^{\ell+1}).$$

By assumption $yf^{(\ell)} = O(x^\ell)$, thus if $\ell \neq h_0 - 2$ we can solve $yf^{(\ell+1)} = O(x^{\ell+1})$ and this uniquely determines $f_{\ell+1}|_\Sigma$.

On the other hand if $\ell = h_0 - 2$ then (5.1) shows that, modulo $O(x^{\ell+1})$,

$$yf^{(\ell)} = y(f^{(\ell)} + x^{\ell+1} f_{\ell+1}),$$

regardless of $f_{\ell+1}$. It follows that the map $f_0 \mapsto x^{-\ell} yf^{(\ell)}$ is tangential and $x^{-\ell} yf^{(\ell)}|_\Sigma$ is the obstruction to solving $yf^{(\ell+1)} = O(x^{\ell+1})$. By a simple induction this is seen to be a non-zero multiple of $y^{\ell+1} f_0|_\Sigma$. \square

Thus by induction we conclude the following.

Proposition 5.4. *For $h_0 \notin \mathbb{Z}_{\geq 2}$ Problem (5.1) can be solved to order $\ell = \infty$. For $h_0 \in \mathbb{Z}_{\geq 2}$ the solution is obstructed by $[P_{h_0-1}f]|_\Sigma$; if, for a particular f , $[P_{h_0-1}f]|_\Sigma = 0$ then there is a family of solutions to order $\ell = \infty$ parametrised by sections $f_{h_0-1} \in \Gamma \mathcal{E}^\Phi[-d - w_0 + 1]|_\Sigma$.*

If (M, c, σ) is almost Einstein then Problem (5.1) can be solved to order $\ell = \infty$ for all $(h_0 + 1) \notin 2\mathbb{Z}_{\geq 2}$.

Note the second part of the Proposition follows from the first using Theorem 4.5.

5.2. The formal solution operator. Now to identify the asymptotics found above we re-examine the extension problem 5.1 and show first that, via the $\mathfrak{sl}(2)$ used above, in the generic case it is captured by a simple ODE problem. This enables us to relate the asymptotics to Bessel functions.

We seek to define a formal operator $:K:$ which will map the “initial data” $f_0|_\Sigma$ to the solution. Thus as a first step we posit

$$(5.2) \quad f = (1 + \alpha_1 xy + \alpha_2 x^2 y^2 + \cdots) f_0 := :K(z): f_0,$$

where $z = xy$, $K(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots \in \mathbb{C}[[z]]$ and $: \cdot :$ indicates a *normal ordering* defined by

$$:z^k: = x^k y^k.$$

Using $[y, x^k] = -x^{k-1} k(h + k - 1)$ we have

$$y :z^k: f_0 = : \left[\left(\frac{d}{dz} z \frac{d}{dz} - (h_0 - 1) \frac{d}{dz} + 1 \right) z^k \right] : y f_0,$$

and so

$$(5.3) \quad y :K(z): f_0 = : (zK''(z) - (h_0 - 2)K'(z) + K(z)) : y f_0.$$

Thus finding $K(z)$, so that $f := :K(z): f_0$ formally solves $I \cdot Df = 0$ for any f_0 , amounts to finding in $\mathbb{C}[[z]]$ a solution to the ordinary differential equation

$$(5.4) \quad zK'' - (h_0 - 2)K' + K = 0.$$

Formally solving this is equivalent to formally solving $z^2 K'' - (h_0 - 2)zK' + zK = 0$ which can be re-expressed as

$$(5.5) \quad (E(E - h_0 + 1) + z)K(z) = 0,$$

with

$$E = z \frac{d}{dz},$$

the Euler operator. Examining this applied to the terms $\alpha_{k-1} z^{k-1} + \alpha_k z^k$ in the expansion of $K(z)$ we immediately recover the equivalent recursion relation

$$(5.6) \quad k(k - h_0 + 1)\alpha_k + \alpha_{k-1} = 0,$$

which is solvable for α_k , $k \in \mathbb{Z}_{\geq 1}$, so long as $h_0 \neq k + 1 = 2, 3, \dots$

In fact up to a simple change of variables and rescaling, the display (5.4) is Bessel's equation. Thus, away from the special weights $h_0 = 2, 3, \dots$ we have¹

$$(5.7) \quad K(z) = z^{\frac{h_0-1}{2}} \Gamma(2 - h_0) J_{1-h_0}(2\sqrt{z}).$$

¹The Bessel function of the first kind $J_\nu(u)$ is defined for complex values u with $|\arg u| < \pi$. For the geometries we consider here, \sqrt{z} — being related to σ and $\frac{-1}{\sqrt{2}} I \cdot D$ — can be purely imaginary, in which case the Bessel function may be rewritten in terms of a modified Bessel function. For simplicity, we work with standard Bessel functions.

The Gamma function is included to achieve the correct normalisation, and the Bessel function of the first kind was chosen as its Taylor series expansion²

$$\begin{aligned} z^{\frac{h_0-1}{2}} \Gamma(2-h_0) J_{1-h_0}(2\sqrt{z}) &= \sum_{m=0}^{\infty} \frac{(-z)^m}{m!(m+1-h_0)_m} \\ &= 1 + \frac{z}{h_0-2} + \frac{z^2}{2(h_0-2)(h_0-3)} + \cdots, \end{aligned}$$

matches the series solutions found above.

Orchestrating the above results we have the following answer to Problem 5.1, away from the exceptional weights.

Proposition 5.5. *On $\mathcal{E}^\Phi[w_0]$, $w_0 \notin \{\frac{j-d}{2} \mid j \in \mathbb{Z}_{\geq 2}\}$, the formal solution operator $:K:$ is given by*

$$f_0 \xrightarrow{:K:} \sum_{m=0}^{\infty} \sigma^m \frac{1}{m!(m+1-h_0)_m (I^2)^m} (I \cdot D)^m f_0.$$

Remark 5.6. Readers familiar with [40] and related treatments of Poincaré–Einstein asymptotics may recall an evenness property of the series expansions there. This feature appears as a consequence of the form of our solutions (see, for example, Proposition 5.5), where every appearance of the scale σ is accompanied by that of the operator $I \cdot D$, both of which, upon making choices corresponding to the setting of [40], yields one power of the coordinate around which their expansion is based.

Remark 5.7. Curiously enough, in a different realisation of the $\mathfrak{sl}(2)$ (as well as for higher rank \mathfrak{g} and super Lie algebras, where xy is replaced by products of generators $x_\alpha y_\alpha$ summed over positive roots α), the same normal ordered expression as $:K:$ (albeit at certain fixed values of h_0) appeared in the completely different contexts of higher spin systems and minisuperspace quantizations of supersymmetric black holes [12, 13, 14, 15, 8]. There the underlying representation of \mathfrak{g} is in terms of algebraic operations on tensors. The normal ordered Bessel function $:K:$ appeared, not as a solution generating operator, but instead was used to construct the differential in a complex whose cohomology described either higher spin excitations or states in a quantized black hole.

At this point it is not yet evident how the formal solution $:K(z):f_0$ depends on the choice of $f_0 \in \mathcal{E}^\Phi[w_0]$ extending $f_0|_\Sigma$. This is settled as follows.

Proposition 5.8. *The solution $:K(z):f_0$ is independent of how f_0 smoothly extends $f|_\Sigma$ off Σ , as a section of $\mathcal{E}^\Phi[w_0]$.*

Proof. Let $f_1 \in \Gamma \mathcal{E}^\Phi[w_0 - 1]$. Then $xf_1 \in \Gamma \mathcal{E}^\Phi[w_0]$ and we consider

$$:z^k:xf_1 = x^k y^k xf_1.$$

Using $[y^k, x] = -y^{k-1}k(h-k+1)$ from (3.1) we see that

$$x^k y^k xf_1 = -x(k(h_0-k-1)x^{k-1}y^{k-1} - x^k y^k)f_1 = x:(k(k+1-h_0)z^{k-1} + z^k):f_1.$$

Thus

$$:K(z):xf_1 = x:(zK'' - (h_0-2)K' + K):f_1 = x:\left[\frac{1}{z}(E(E-h_0+1) + z)K(z)\right]:f_1,$$

²The Pochhammer symbol $(k)_l = k(k-1)\cdots(k-l+1)$ and $(k)_0 = 1$.

and we note the equation (5.4) has again arisen. Since $K(z)$ is a solution of this, the right-hand-side of the display vanishes. \square

In the following it will be convenient to annotate $K(z)$ with h_0 so that $K^{h_0}(z)$ indicates the series in $\mathbb{C}[[z]]$ giving the solution operator on $\mathcal{E}^\Phi[w_0]$ (where $h_0 = d + 2w_0$ as usual).

5.3. Solutions of the second kind. The equation (5.4), being second order, admits a second independent homogeneous solution. This corresponds to another solution of the $I \cdot D$ equation above, but to see this rigorously we first generalise (*cf.* Problem 5.1) our notion of solution as follows.

Problem 5.9. Given $\bar{f}_0|_\Sigma \in \Gamma\mathcal{E}^\Phi[w_0 - \alpha]|_\Sigma$ and an arbitrary extension \bar{f}_0 of this to $\Gamma\mathcal{E}^\Phi[w_0 - \alpha]$ over M , find $\bar{f}_i \in \mathcal{E}^\Phi[w_0 - \alpha - i]$ (over M), $i = 1, 2, \dots$, so that

$$(5.8) \quad \bar{f} := \sigma^\alpha(\bar{f}_0 + \sigma \bar{f}_1 + \sigma^2 \bar{f}_2 + \dots + O(\sigma^{\ell+1}))$$

solves $I \cdot D \bar{f} = O(\sigma^{\ell+\alpha})$, off Σ , for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

As usual we shall write $I \cdot D \bar{f} = 0$ as a shorthand for the statement that \bar{f} is an $\ell = \infty$ solution.

The exponent α , if not integral, takes Problem 5.9 outside the realm of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and its modules. However $\sigma = x \in \mathfrak{g}$ is a smooth section of an oriented real line bundle and as such x^α is well defined for any complex α . What is more, as follows from Remark 3.2, within the operator algebra Corollary 3.5 allows the extension

$$(5.9) \quad [x^\alpha, y] = x^{\alpha-1} \alpha (h + \alpha - 1).$$

Since \bar{f}_0 may be non-vanishing along Σ it follows immediately from (5.9) that $I \cdot D \bar{f} = 0$ has no solution unless

$$\alpha = 0 \quad \text{or} \quad \alpha = h_0 - 1,$$

where $h\bar{f} = h_0\bar{f}$. Since $\alpha = 0$ is the case of the extension problem treated above, we set here $\alpha := h_0 - 1$. Let us write \bar{h}_0 for the h -weight of \bar{f}_0 . Observe that since $h(x^\alpha \bar{f}_0) = h_0(x^\alpha \bar{f}_0)$, it follows that

$$1 - \bar{h}_0 = h_0 - 1.$$

Now write

$$G(z) = 1 + \beta_1 z + \beta_2 z^2 + \dots$$

for the element of $\mathbb{C}[[z]]$ so that

$$:G(z): = 1 + \beta_1 xy + \beta_2 x^2 y^2 + \dots,$$

and $(x^\alpha :G(z):) : \mathcal{E}^\Phi[w_0 - \alpha] \rightarrow \mathcal{E}^\Phi[w_0]$ is the putative solution operator (with $:$ the normal ordering as before). Following the ideas above we now seek an ODE that determines G . But note that since $\alpha = h_0 - 1$ then, on $\mathcal{E}^\Phi[w_0 - \alpha]$, we have $y x^\alpha :G(z): = x^\alpha y :G(z):$. Thus, upon the replacement of h_0 with \bar{h}_0 , we are in fact reduced to *the same* ODE

$$(5.10) \quad (zG')' - (\bar{h}_0 - 1)G' + G = 0 \quad \Leftrightarrow \quad (E(E - \bar{h}_0 + 1) + z)G(z) = 0,$$

as in (5.4). It follows that $G(z)$ is given by (5.7) with this replacement, that is

$$(5.11) \quad G(z) = K^{\bar{h}_0}(z) = z^{\frac{\bar{h}_0-1}{2}} \Gamma(2 - \bar{h}_0) J_{1-\bar{h}_0}(2\sqrt{z}) = z^{\frac{1-h_0}{2}} \Gamma(h_0) J_{h_0-1}(2\sqrt{z}).$$

Remark 5.10. Using $Ez^\alpha = z^\alpha(E + \alpha)$, It is easily verified that $z^\alpha G(z)$ is a (“second”) solution to (5.4).

For the record we note that, in terms of the weight h_0 , the ODE (5.10) is

$$(5.12) \quad (zG')' + (h_0 - 1)G' + G = 0.$$

Expressed alternatively, we wish to solve

$$(E(E + h_0 - 1) + z)G(z) = 0,$$

which leads to the equivalent recursion relation

$$k(k + h_0 - 1)\beta_k + \beta_{k-1} = 0.$$

This is solvable for β_k , $k \in \mathbb{Z}_{\geq 1}$, now with the restriction $h_0 \neq 1 - k = 0, -1, \dots$

Putting together the results from Section 5.2 and Section 5.3 we summarise as follows.

Proposition 5.11. *Problem 5.9 has a solution only if $\alpha = 0$ or $\alpha = h_0 - 1$. If $\alpha = 0$ and $h_0 \notin \mathbb{Z}_{\geq 2}$ then an infinite order solution is given by*

$$(5.13) \quad f_0 \mapsto :K^{h_0}:f_0,$$

where f_0 is any section of $\mathcal{E}^\Phi[w_0]$ smoothly extending $f_0|_\Sigma \in \mathcal{E}^\Phi[w_0]|_\Sigma$.

If $\alpha = h_0 - 1$ and $h_0 \notin \mathbb{Z}_{\leq 0}$ then an infinite order solution is given by

$$(5.14) \quad \bar{f}_0 \mapsto x^{h_0-1}:K^{\bar{h}_0}:\bar{f}_0,$$

where \bar{f}_0 is any smooth extension of $\bar{f}_0|_\Sigma$ to a section of $\mathcal{E}^\Phi[-d - w_0 + 1]$.

In each case the solution is independent of how f_0 (respectively \bar{f}_0) extends $f|_\Sigma$ (respectively $\bar{f}|_\Sigma$) off Σ . Thus (5.13) and (5.14) determine well-defined solution operators

$$(:K^{h_0}:): \Gamma\mathcal{E}^\Phi[w_0]|_\Sigma \rightarrow \Gamma\mathcal{E}^\Phi[w_0],$$

and

$$(x^{h_0-1}:K^{\bar{h}_0}:): \Gamma\mathcal{E}^\Phi[w_0 - h_0 + 1]|_\Sigma \rightarrow \Gamma\mathcal{E}^\Phi[w_0].$$

For $h_0 \notin \mathbb{Z}$ the solutions (5.13) and (5.14) are distinct if $f_0|_\Sigma$ and $\bar{f}_0|_\Sigma$ are not zero. Further any formal solution $f \in \Gamma\mathcal{E}^\Phi[w_0]$ of the equation $I \cdot Df = 0$ (off Σ) is a linear combination of solutions of the form (5.13) and (5.14).

Remark 5.12. When h_0 is a positive integer the second solution (5.14) still provides an infinite order solution to Problem 5.9. Moreover, this is an instance of the situation where one can compose tangential operators: for positive integer h_0 , the second solution also furnishes a solution to a modified version of Problem 5.9

$$f_0 \mapsto x^{h_0-1}:K^{\bar{h}_0}:y^{h_0-1}f_0,$$

where instead of $\bar{f}_0|_\Sigma$, one is given an $f_0|_\Sigma \in \Gamma\mathcal{E}^\Phi[w_0]|_\Sigma$ and an arbitrary extension f_0 of this to $\Gamma\mathcal{E}^\Phi[w_0]$ over M . Again, this solution is independent of how f_0 extends $f_0|_\Sigma$ off Σ .

Finally, note that $h_0 = 1$ (i.e. $w_0 = -\frac{d-1}{2}$) is allowed for both (5.13) and (5.14), but in this case they coincide. This amounts to the situation of a repeated root of the indicial equation for the Frobenius method for second order ODEs. This suggests a new second solution, perhaps of the form $\bar{F}(\sigma) \log \sigma$, where $\bar{F}(\sigma)$ is again a power series type solution. The weights $h_0 \in \mathbb{Z} \setminus \{1\}$ should also be dealt with by log terms. With σ a section of $\mathcal{E}_+[1]$ we can make sense of $\log \sigma$ using the notion of log-densities developed in Section 2.1.

5.4. Solutions with log terms. We now consider the case of weights $h_0 \in \mathbb{Z}$. From above we see that for each such weight there is just one smooth solution, *i.e.* solution to Problem 5.9. In view of the symmetry between the cases of weights h_0 and \bar{h}_0 it suffices to treat the case of $h_0 \in \mathbb{Z}_{\geq 1}$.

We shall assume that (M, c, σ) is conformally compact with $\sigma \geq 0$. Recalling the discussion of log density bundles from Section 2.1 we note that $\log \sigma$ is then well defined on the interior of M , where $\sigma > 0$, and we include $\log \sigma = \log x$ into our formal computations. Since we understand pointwise the meaning of complex linear operations on log densities we can take the tensor product of $\mathcal{F}[w]$ with vector bundles, and in particular weighted tractor bundles.

A slight subtlety arises at this point, because $\log \sigma$ is not an eigen-section for the weight operator, *cf.* (2.3). Thus we introduce a second scale: let τ denote some true scale on M , *i.e.* $\tau \in \Gamma \mathcal{E}[1]$ is nowhere zero on M ; in particular $\tau|_{\Sigma}$ gives a scale on Σ , and we may write g^τ for the metric $\tau^{-2}g$ on M determined by τ .

Problem 5.13. Let $h_0 = d + 2w_0$ be in $\mathbb{Z}_{\geq 2}$. Given non-zero $f_0|_{\Sigma} \in \Gamma \mathcal{E}^\Phi[w_0]|_{\Sigma}$ and an arbitrary extension f_0 of this to $\Gamma \mathcal{E}^\Phi[w_0]$ over M , find $f_i \in \mathcal{E}^\Phi[w_0 - i]$, $i = 1, 2, \dots$, and $\bar{f}_j \in \mathcal{E}^\Phi[-d - w_0 + 1 - j]$ (over M), $j = 0, 1, \dots$, so that

$$(5.15) \quad f := (f_0 + \sigma f_1 + \dots) + \sigma^{h_0-1}(\log \sigma - \log \tau)(\bar{f}_0 + \sigma \bar{f}_1 + \dots) + O(\sigma^{\ell+1})$$

solves $I \cdot Df = O(\sigma^\ell)$, off Σ , for $\ell \in \mathbb{N} \cup \infty$ as high as possible. More precisely, if $\ell \geq h_0 - 1$ then the display should also include $+O(\sigma^{\ell+1} \log \sigma)$, meaning up to the addition of a formally smooth term tensor product with $\sigma^{\ell+1} \log \sigma$, and then we seek to solve $I \cdot Df = O(\sigma^\ell) + O(\sigma^\ell \log \sigma)$, for ℓ as high as possible.

If $h_0 = 1$ we seek instead a solution with $\bar{f}_0|_{\Sigma}$ not zero and taken as the initial data, and $f_0 = 0$.

Remark 5.14. Three comments on the statement of Problem 5.13 are important:

First note that we have explicitly entered a second scale, *viz.* the scale τ that extends to the boundary. Although we may treat the $I \cdot D$ equation without doing this, the manoeuvre is necessary if we require that f is an eigenfunction of the weight operator (enabling the interior relation (2.10)). Note that $\log \sigma - \log \tau = \log r$ is a function.

Second we have decreed where the log terms enter by adding a term of the form $\sigma^{h_0-1} \log \sigma \bar{f}$. Alternatively we could have added $\sigma^\beta \log \sigma \bar{f}$ for some β . It is straightforward to then show that $\beta = h_0 - 1$ is forced.

Third, in the statement of the problem we have stipulated that $\log \tau$ appear in the combination $\log \sigma - \log \tau$. Although we cannot relax this requirement, we shall see that the simplest presentation of the solution is not *explicitly* in this form, but differs by terms involving various reorderings of the operators y and $\log \tau$.

Remark 5.15. Allowing $f_0 = 0$, Problem 5.13 always has a solution of the second kind (discussed in Section 5.3) with the leading power of σ equaling the integer $h_0 - 1$. It is given by the formula $x^{h_0-1}; K^{2-h_0}(z); f_{h_0-1}$ where $f_{h_0-1}|_{\Sigma} \in \Gamma \mathcal{E}^\Phi[-d - w_0 + 1]|_{\Sigma}$ and f_{h_0-1} is an arbitrary extension of this to $\Gamma \mathcal{E}^\Phi[-d - w_0 + 1]$ over M . In the following discussion, we focus on the case of vanishing f_{h_0-1} ; this term will be supplanted by the leading coefficient of the logarithm \bar{f}_0 (which is of the same weight as f_{h_0-1}).

Let us write

$$f := f_0 + x f_1 + x^2 f_2 + \dots$$

and

$$\bar{f} := \bar{f}_0 + x \bar{f}_1 + x^2 \bar{f}_2 + \cdots .$$

Thus we seek a solution of the form

$$(5.16) \quad f + x^{h_0-1} (\log x - \log \tau) \bar{f} .$$

To determine the equations on f and \bar{f} we need to draw $\log x$ into our operator algebra. Formally, the identity

$$\lim_{k \rightarrow 0} \frac{x^k - 1}{k} = \log x$$

coupled with the relations $[h, x] = 2x$ and $[y, x^k] = -x^{k-1}k(h+k-1)$ from Proposition 3.4 and Corollary 3.5 suggest that

$$(5.17) \quad [h, \log x] = 2, \quad [y, \log x] = -x^{-1}(h-1) .$$

The first of these relations does hold rather generally; if $\mu \in \Gamma\mathcal{E}_+[w_0]$ is any positive weighted conformal density, then $\log \mu$ is a weighted log density, *viz.* a section of $\mathcal{F}[w_0]$ and, by virtue of (2.3) obeys

$$[h, \log \mu] = 2w_0$$

read as an operator relation acting on any section of a weighted tractor bundle. The second relation in (5.17) can be easily proven using ambient tractor machinery, as provided by Proposition 6.1. It also holds on arbitrary sections of weighted tractor bundles. In fact both relations in (5.17) can be shown (again by ambient techniques) to hold also on log densities or tensor products of these with conformal densities, or more generally on a log density tensor product with any weighted tractor bundle. In the remainder of this Section we will develop various consequences of the above identities, with the same range of validity, and state them without explicit reference to the class of sections on which they act.

Now we apply y to (5.16). Since the weight of \bar{f} is such that $[y, x^{h_0-1}]\bar{f} = 0$, and similarly $[y, x^{h_0-1}](\log x - \log \tau)\bar{f} = 0$ new terms arise only from the commutator $[y, \log x - \log \tau]$. Thus the equation from y applied to (5.16) is

$$(5.18) \quad 0 = yf - [1 - h_0]x^{h_0-2}\bar{f} + x^{h_0-1}(\log x y - y \log \tau)\bar{f} .$$

Some observations greatly simplify the analysis of this. First it is not difficult to see that the $\log x$ terms must vanish separately and so

$$y\bar{f} = 0 .$$

Given the weight of \bar{f} we know that there is a formal power series type solution of the form $\bar{f} = :\bar{F}(z): \bar{f}_0$, so we obtain the equation on $\bar{F}(z)$ from (5.12):

$$(5.19) \quad [E(E + h_0 - 1) + z] \bar{F}(z) = 0 .$$

From Section 5.3 this is solvable for $h_0 \in \mathbb{Z}_{\geq 1}$, yielding

$$(5.20) \quad \bar{F}(z) = K^{\bar{h}_0}(z) .$$

At this point our proposed solution (5.16) reads

$$f + x^{h_0-1} (\log x - \log \tau) :K^{\bar{h}_0}(z): \bar{f}_0$$

and we have as yet not determined f or \bar{f}_0 . Our second observation is that we may modify the expression (5.16) and seek a formal solution of the form

$$(5.21) \quad f + x^{h_0-1} (\log x :K^{\bar{h}_0}(z): - :K^{\bar{h}_0}(z): \log \tau) \bar{f}_0 ,$$

because moving $\log \tau$ past the operator $:K^{\bar{h}_0}(z):$ produces terms, each of a well-defined weight, which can be absorbed into the as yet undetermined \mathbf{f} at orders higher than x^{h_0-1} . The utility of this manoeuvre will soon be clear.

Next we upgrade the identity (5.3) to an operator relation for formal solution operators $:K(z): \in \mathbb{C}[[z]]$,

$$(5.22) \quad y :K(z): = : (zK''(z) + 2K'(z) + K(z)) : y - :K'(z): y h.$$

Using this, as well as (2.3) and the fact that (5.20) solves (5.19), we find that the equation arising from the modified expression for our proposed solution (5.21) is

$$(5.23) \quad y \mathbf{f} = \left[1 - h_0\right] x^{h_0-2} :K^{\bar{h}_0}(z): \bar{f}_0 - 2x^{h_0-1} : \frac{dK^{\bar{h}_0}(z)}{dz} : y \bar{f}_0.$$

Notice that for any \bar{f}_0 all log terms have dropped out leaving an inhomogeneous equation for \mathbf{f} . Settling this up to terms of order x^{h_0-2} is the problem

$$(5.24) \quad y \mathbf{f} = \left[1 - h_0\right] x^{h_0-2} :K^{\bar{h}_0}(z): \bar{f}_0 + O(x^{h_0-1}).$$

Here we see that there are two cases $h_0 = 1$, and $h_0 \in \mathbb{Z}_{\geq 2}$. We treat the latter first.

For $h_0 \in \mathbb{Z}_{\geq 2}$, and $f_0|_{\Sigma}$ non-zero, the equation (5.24) can be treated in two stages. The first of these consists of solving $y \mathbf{f} = O(x^{h_0-2})$. This is simply an initial finite part of the Problem 5.1 treated in Section 5.1. Thus we obtain a solution (the unique solution to the given order, if $f_0|_{\Sigma}$ is assumed not zero) $\mathbf{f} = :F_{h_0-2}(z): f_0 + O(x^{h_0-2})$ where the *polynomial* $F_{h_0-2}(z)$, arising as the operator expressing the partial sum up of terms up to order x^{h_0-2} in \mathbf{f} , obeys

$$[E(E - h_0 + 1) + z] F_{h_0-2}(z) = O(z^{h_0-2}).$$

Explicitly

$$(5.25) \quad F_{h_0-2}(z) = 1 + \frac{z}{h_0 - 2} + \frac{z^2}{2!(h_0 - 2)(h_0 - 3)} + \cdots + \frac{z^{h_0-2}}{((h_0 - 2)!)^2},$$

which amounts to solving (5.6) when $k = 1, 2, \dots, h_0 - 2$. So

$$(5.26) \quad y :F_{h_0-2}(z): f_0 = \frac{1}{((h_0 - 2)!)^2} x^{h_0-2} y^{h_0-1} f_0.$$

Now inspecting (5.6) we see the following: the equation $: [E(E - h_0 + 1) + z] F(z) : f_0 = 0$ cannot be solved to higher order in general, with the tangential operator $y^{h_0-1} f_0 = P_{h_0-1} f_0$ (of Theorem 4.1) the obstruction (if this vanishes we may continue, cf. Lemma 5.3 and Proposition 5.4). At the next order \bar{f}_0 provides the rescue in (5.24), in that (5.24) can be solved by setting

$$(5.27) \quad \bar{f}_0 = -\frac{1}{(h_0 - 1)!(h_0 - 2)!} y^{h_0-1} f_0.$$

Equation (5.23) is now solved canonically to all higher orders as follows. We extend $\mathbf{f} = :F_{h_0-2}(z): f_0 + O(x^{h_0-1})$ by first setting the undetermined coefficient of x^{h_0-1} to zero. (Note that any other choice is tantamount to adding to our solution a non-trivial solution of the second kind, as discussed in Remark 5.15.) By (5.27) \bar{f}_0 is determined by f_0 , whence so the right-hand-side of (5.23) as a power series type expansion in x . Now in equation (5.23) we have dealt with all terms below order x^{h_0-1} . But after this the equation is solvable recursively, since it is simply an inhomogeneous version of the

recursion relation (5.6) of the form $k(k - h_0 + 1)\alpha_k + \alpha_{k-1} = r_{k-1}$, $k = h_0, h_0 + 1, \dots$, where r_k indicates terms corresponding to the right-hand-side of (5.23).

In fact, this last problem can be neatly formulated in terms of an inhomogeneous ordinary differential equation along the lines of (5.4). This surprisingly enables a complete and explicit solution to the problem. In total the treatment involves several subtleties; to reveal and deal with each of these we break the treatment into a number of stages. The first step is that we express also the $O(x^{h_0})$ terms in \mathbf{f} in the solution generating operator language

$$(5.28) \quad \mathbf{f} = :F_{h_0-2}(z): f_0 + x^{h_0} :B(z): y \bar{f}_0,$$

where it now only remains to determine $B(z) \in \mathbb{C}[[z]]$ (here \bar{f}_0 is given as in (5.27)). Inserting this, along with (5.26) and (5.27) into our equation (5.23) gives

$$x^{h_0-2} \left((1 - h_0) + xy :zB(z): \right) \bar{f}_0 = x^{h_0-2} : \left((1 - h_0) K^{\bar{h}_0}(z) - 2z \frac{dK^{\bar{h}_0}(z)}{dz} \right) : \bar{f}_0.$$

Here we used Corollary (3.5) and the fact that $\bar{f}_0 \in \mathcal{E}^\Phi[-d - w_0 + 1]$, along with the simple operator identity $x :K(z): y = :zK(z):$ for any $K(z) \in \mathbb{C}[[z]]$. We can solve this equation using the formal solution operator methods explained in Section 5.1 (and in particular equation (5.3)). This gives the inhomogeneous, linear ODE

$$(5.29) \quad (E(E + h_0 - 1) + z)[zB(z)] = -(2E + h_0 - 1)[K^{\bar{h}_0}(z) - 1].$$

Clearly the Frobenius method yields a unique formal solution for $B(z)$, the leading terms of which we explicate below

$$\begin{aligned} B(z) = & \frac{h_0+1}{h_0^2} - \frac{(h_0+2)(3h_0+1)}{4h_0^2(h_0+1)^2} z + \frac{(h_0+3)(11h_0^2+18h_0+4)}{36h_0^2(h_0+1)^2(h_0+2)^2} z^2 \\ & - \frac{(h_0+4)(25h_0^3+98h_0^2+99h_0+18)}{288h_0^2(h_0+1)^2(h_0+2)^2(h_0+3)^2} z^3 + \dots \end{aligned}$$

The solution to all orders for $B(z) = \sum_{k=0}^{\infty} \gamma_k z^k \in \mathbb{C}[[z]]$ is given by the recurrence

$$\begin{aligned} \gamma_0 &= \frac{h_0 + 1}{h_0^2}, \\ \gamma_k &= -\frac{1}{(k+1)(h_0+k)} \left[\gamma_{k-1} + \frac{(-1)^{k+1}(h_0+2k+1)}{(k+1)!(h_0+k)_{k+1}} \right]. \end{aligned}$$

Remark 5.16. Use of the computer package Maple gives the closed-form solution to this recursion:

$$\gamma_k = \frac{(-1)^k \left(h_0 \sum_{j=0}^{k-1} \frac{h_0+2j+3}{(j+2)(h_0+j+1)} + h_0 + 1 \right) (h_0 - 1)!}{h_0(h_0+k)!(k+1)!}.$$

Putting together the polynomial $F_{h_0}(z)$ and power series $B(z)$, $K^{\bar{h}_0}(z) \in \mathbb{C}[[z]]$ (given by equations (5.25), (5.29) and (5.11), respectively) we may now build a solution operator $\hat{\mathcal{O}}$. Acting on sections of $\mathcal{E}^\Phi[w_0]$, this is given by

$$(5.30) \quad \begin{aligned} \hat{\mathcal{O}} &= :F_{h_0-2}(z): - \frac{:z^{h_0} B(z):}{(h_0-1)!(h_0-2)!} \\ &- \frac{x^{h_0-1} (\log x :K^{\bar{h}_0}(z): - :K^{\bar{h}_0}(z): \log \tau) y^{h_0-1}}{(h_0-1)!(h_0-2)!}. \end{aligned}$$

By construction this has the required property

$$(5.31) \quad y \hat{\mathcal{O}} f_0 = 0$$

for any section f_0 of $\mathcal{E}^\Phi[w_0]$. However, we have not yet reached the ideal solution to the problem (5.13). The point here is that (5.30) is not independent of how f_0 is extended off Σ from given boundary data $f_0|_\Sigma \in \Gamma\mathcal{E}^\Phi[w_0]|_\Sigma$. This statement is readily verified by explicitly computing $\hat{\mathcal{O}}xf_1$ for some $f_1 \in \Gamma\mathcal{E}^\Phi[w_0 - 1]$

$$(5.32) \quad \hat{\mathcal{O}}xf_1 = \frac{x : z^{h_0-2} K^{\bar{h}_0}(z) : f_1}{((h_0 - 2)!)^2}.$$

The last computation relied on the second relation in Corollary 3.5 as well as analog of the identity (5.22), adapted to the case where the operator x appears on the right, rather than y on the left, namely

$$(5.33) \quad :K(z):x = x:(zK''(z) + K(z)):-x:K'(z):h.$$

There is however, no analog of the second relation in (5.17) for the commutator of the operator y with $\log x$, because x and $\log \tau$ simply commute. This absence precisely explains the leftover terms in (5.32) whilst at same time suggests a possible remedy. Examining the operator $\hat{\mathcal{O}}$ given above, we see that there is a complete symmetry between the operator x on the left and y on the right (remembering that $:z^k: = x^k y^k$) except for where $\log \tau$ appears in the place where symmetry would have suggested a term “ $-\log y$ ”. Remarkably it is possible to adjust (5.30) in such a way that $-\log \tau$ does effectively play the *rôle* of the logarithm of y for the algebraic purposes required. Remembering that we may always add some amount of a solution of the second kind, to the one obtained already in (5.30), we find that gives us sufficient leeway to perform such a manoeuvre. Our cure begins by observing that for any section \bar{f}_0 of $\mathcal{E}^\Phi[-d - w_0 + 1]$ (so that $h\bar{f}_0 = (2 - h_0)\bar{f}_0 = \bar{h}_0\bar{f}_0$) we have the identity

$$(5.34) \quad [y, x^{h_0-1} \log x] \bar{f}_0 = (1 - h_0) x^{h_0-2} \bar{f}_0,$$

which was obtained using Corollary 3.5 and the algebra of y with $\log x$ in (5.17). What we would like is a close analog of this result for commutators of the operator x with $\log \tau$ and powers of y . While *a priori* the existence of such is not at all guaranteed, it turns out that the Weyl averaged operator

$$(5.35) \quad (\log \tau y^{h_0-1})_W := \frac{\log \tau y^{h_0-1} + y^{h_0-1} \log \tau}{2}, \quad h_0 \in \mathbb{Z}_{\geq 1},$$

obeys

$$(5.36) \quad \left[(\log \tau y^{h_0-1})_W, x \right] f_1 = (1 - h_0) y^{h_0-2} f_1,$$

where f_1 is any section of $\mathcal{E}^\Phi[w_0 - 1]$ (so that $hf_1 = (h_0 - 2)f_1$). The validity of the identity (5.34) is essentially the reason why the introduction of the logarithm of x enabled us to write the solution operator $\hat{\mathcal{O}}$ in (5.30) annihilated by the action of y from the left. The new identity (5.36) exactly mimics that result, for the case now of x acting from the right. This leads us to the new solution generating operator:

$$(5.37) \quad \begin{aligned} \mathcal{O} &= :F_{h_0-2}(z): - \frac{:z^{h_0} B(z):}{(h_0 - 1)!(h_0 - 2)!} \\ &- \frac{x^{h_0-1} \log x :K^{\bar{h}_0}(z): y^{h_0-1} - x^{h_0-1} :K^{\bar{h}_0}(z): (\log \tau y^{h_0-1})_W}{(h_0 - 1)!(h_0 - 2)!}. \end{aligned}$$

It is then a matter of straightforward algebra, using the relations (5.34), or (5.36), along with (5.22), or (5.33), to establish the critical properties

$$y\mathcal{O}f_0 = 0, \quad \text{and} \quad \mathcal{O}xf_1 = 0,$$

for arbitrary sections f_0 and f_1 of $\mathcal{E}^\Phi[w_0]$ and $\mathcal{E}^\Phi[w_0 - 1]$, respectively. In fact for the case of verifying $y\mathcal{O}f_0 = 0$, this also follows from (5.31), since by construction $\widehat{\mathcal{O}}f_0$ and $\mathcal{O}f_0$ differ by a solution of the second kind. For the case of showing $\mathcal{O}xf_1 = 0$ we follow the idea of Proposition 5.8, although the situation here is significantly more delicate: It is straightforward to verify that, through the judicious introduction of the Weyl averaged expression (5.35) in (5.37), we are again led to the equation (5.29) on $zB(z)$; the equation which $B(z)$ solves, by its definition. At this juncture, we also note that an easy variant of the computation directly above shows that replacing $\log \tau$ by $\log \tau + xt_1$ where t_1 is any section of $\mathcal{E}[-1]$ does not change the solution $\mathcal{O}f_0$. In other words, the solution only depends on initial data $\tau|_\Sigma$ and is independent of the choice of extension of this to $\mathcal{E}[1]$ over M .

Remark 5.17. One might wonder whether, by making a distinguished choice of second scale τ , it could be possible to avoid introducing the Weyl ordered combination of operators y and $\log \tau$ defined in (5.35). In fact it is straightforward to prove that y and $\log \tau$ do not commute when I^2 is non-vanishing.

Now for the case $h_0 = 1$: We again search for a formal solution of the form (5.21) with $h_0 = 1$, but recall in this instance we take $\overline{f}_0|_\Sigma$ as the given initial data. This is required by (5.23) since the first term on the right-hand-side now vanishes. We determine directly \overline{f} in terms of \overline{f}_0 using (5.20). Then we write \overline{f} as in (5.28) but without the first term since $f_0 = 0$ here. Thus the expansion of \overline{f} is given in terms of $B(z)$, which (as for higher h_0 cases) is determined by the inhomogeneous ODE (5.29). To ensure that the solution is independent of how $\overline{f}_0|_\Sigma$ is extended off Σ we use the same Weyl averaging technique as above. Thus, in this case, the solution generating operator is given by the numerators of the second and third terms in (5.37), evaluated at $h_0 = 1$.

We have by now established the following

Theorem 5.18. *Problem 5.13 has a canonical infinite order solution with $f_{h_0-1} = 0$ when $1 \neq h_0 \in \mathbb{Z}_{\geq 2}$*

$$(5.38) \quad (f_0, \tau) \mapsto \mathcal{O}f_0,$$

where f_0 and τ are, respectively, any sections of $\mathcal{E}^\Phi[w_0]$ and $\mathcal{E}[1]$ (the solution operator \mathcal{O} is given explicitly in (5.37)).

If $h_0 = 1$, then an infinite order solution is given by

$$(5.39) \quad (\overline{f}_0, \tau) \mapsto \overline{\mathcal{O}}\overline{f}_0,$$

where \overline{f}_0 and τ are, respectively, any sections of $\mathcal{E}^\Phi[\frac{1-d}{2}]$ and $\mathcal{E}[1]$. The solution operator $\overline{\mathcal{O}}$ is

$$\overline{\mathcal{O}} = \log x :J_0(2\sqrt{z}): - :J_0(2\sqrt{z}): \log \tau + :zB(z):.$$

In each case the solution $\mathcal{O}f_0$ (respectively $\overline{\mathcal{O}}\overline{f}_0$) is independent of how τ as well as f_0 (respectively \overline{f}_0) extend $\tau|_\Sigma$ and $f|_\Sigma$ (respectively $\overline{f}|_\Sigma$) off Σ and thus determine well-defined solution operators

$$\mathcal{O} : (\Gamma\mathcal{E}^\Phi[w_0]|_\Sigma, \Gamma\mathcal{E}[1]|_\Sigma) \rightarrow \Gamma\mathcal{E}^\Phi[w_0],$$

and

$$\overline{\mathcal{O}} : \left(\Gamma \mathcal{E}^\Phi \left[\frac{1-d}{2} \right] |_\Sigma, \Gamma \mathcal{E}[1] |_\Sigma \right) \rightarrow \Gamma \mathcal{E}^\Phi \left[\frac{1-d}{2} \right].$$

Remark 5.19. The case of (M, c, σ) being Poincaré-Einstein. For Problem 5.13 we found that the coefficient of $\log \sigma$ satisfies

$$\overline{f}_0 = -\frac{1}{(h_0 - 1)!(h_0 - 2)!} P_{h_0-1} f_0$$

where P_k is the tangential operator from Theorem 4.1. Any other solution of the $I \cdot Df = 0$ equation, is a linear combination of this with a solution of Problem 5.9.

It is interesting to note the specialisations that arise if (M, c, σ) is a Poincaré-Einstein structure, which we now assume here. Then the scale tractor I is parallel, and the interior is Einstein. This means that on the interior (M_+, c) the powers of $I \cdot D$ give the *interior GJMS operators*, when acting on densities of the appropriate weight. This features precisely as follows.

Suppose that $j \in \mathbb{Z}_{\geq 1}$ and $h_0 = 2j$ and so f_0 has conformal weight $w_0 = j - \frac{d}{2}$. Using that (M, c, σ) is conformally Einstein, from [26] we have that $y^j f_0 = x^j Y_j f_0$ where (up to a constant) Y_j is the order $2j$ GJMS operator (in general in fact the generalisation thereof to a tractor twisted version) on (M, c) . Thus $y^{h_0-1} f_0 = y^{j-1} x^j Y_j f_0$ and using (3.1) this is $O(x)$. Thus by (5.27) $\overline{f}_0|_\Sigma = 0$ and by Proposition 5.8, also $:K^{\overline{h}_0} : \overline{f}_0 = 0$. Thus the canonical solution has no log terms.

In summary, if (M, g^o) is Poincaré-Einstein then:

- If $h_0 = 3, 5, 7, \dots$, then P_{h_0-1} is the (tractor twisted) GJMS operator.
- If $h_0 = 2, 4, 6, \dots$, then P_{h_0-1} is trivial, $\overline{f}_0 = 0$ and there are no log terms in the canonical solution.

Finally recall that we introduced the scale choice τ so that we could describe a solution with a well-defined conformal weight, despite the presence of log terms. Having solved the problem it is most natural to finally express the answer in the scale determined by τ , that is in terms of the metric $g^\tau := \tau^{-1} g$. In this trivialisation of the weight bundles, τ itself is represented by the constant function 1. Then since $\log 1 = 0$ the expression, in this scale, for the solution is obtained from that found above by simply formally replacing σ and its powers by $r = \sigma/\tau$ and its powers.

5.5. The log density problem. We now consider the problem of formally solving

$$I \cdot D U = 0$$

for a log density $U \in \Gamma \mathcal{F}[1]$. We again treat a Dirichlet type problem where $U|_\Sigma$ is the initial data. Given an arbitrary smooth extension $U_0 \in \Gamma \mathcal{F}[1]$ of $U|_\Sigma$ we study the following problem:

Problem 5.20. Given $U_0|_\Sigma$ and an arbitrary extension of this to $\mathcal{F}[1]$ over M , find $U_i \in \mathcal{E}[-i]$ (over M , $i \in \mathbb{Z}_{\geq 1}$), such that

$$U^{(l)} := U_0 + \sigma U_1 + \sigma^2 U_2 + \dots + O(\sigma^{l+1})$$

solves $I \cdot D U = O(\sigma^l)$, off Σ , for $l \in \mathbb{N} \cup \infty$ as high as possible.

Remark 5.21. Since the sum of a log density and weight 0 conformal density (*i.e.*, a function) is again a log density, all terms in the expansion U_i with $i \geq 1$ are taken to be conformal densities with weight such that the product $\sigma^i U_i$ is a weight zero conformal density. Moreover, since $I \cdot D U_0$ is a weight -1 conformal density (even though U_0 is a

log density), the solution to the above problem can be achieved by applying the results developed in previous sections for conformal densities, in particular those for $h_0 = d$ (because then $w_0 = 0$).

In light of the above remark, from Theorem 5.18 we directly obtain the following

Corollary 5.22. *Problem 5.20 has a canonical infinite order solution with*

$$(U_0, \tau) \mapsto \mathcal{O}|_{h_0=d} f_0,$$

where U_0 and τ are, respectively, any sections of $\mathcal{F}[1]$ and $\mathcal{E}[1]$ and the solution operator $\mathcal{O}|_{h_0=d}$ is precisely the one given in (5.37) save that h_0 is set to the value $h_0 = d$.

The solution $\mathcal{O}|_{h_0=d} U_0$ is independent of how τ as well as U_0 extends $f|_\Sigma$ (respectively $\bar{f}|_\Sigma$) off Σ and thus determines a well-defined solution operator

$$\mathcal{O}|_{h_0=d} : (\Gamma\mathcal{F}[1]|_\Sigma, \Gamma\mathcal{E}[1]|_\Sigma) \rightarrow \Gamma\mathcal{F}[1].$$

Remark 5.23. In the spirit of Remark 5.19 above some comments, extending the Corollary, are worthwhile. These again use Theorem 5.18.

First note that, in the solution, the coefficient of $\log \sigma$ is

$$(5.40) \quad -\frac{1}{(n)!(n-1)!} P_n U_0$$

where P_n is the tangential operator from Theorem 4.1. It is natural to consider this in the case that $U_0 = -\log \tilde{\mu}$, where $\tilde{\mu}$ is some extension to $\Gamma\mathcal{E}_+[1]$ of a section $\mu \in \Gamma\mathcal{E}_+[1]|_\Sigma$ so that $U_0 \in \mathcal{F}[-1]$. (Recall that for log densities pointwise multiplication is a bundle isomorphism, so here we are simply multiplying the entire solution by a factor -1 .)

Then specialising to the case of (M, c, σ) Poincaré-Einstein we have $I^2 = 1$, and from (2.11) we see that in the interior scale the problem being solved is

$$(5.41) \quad -\Delta^{g^o} U = n,$$

In this case, from Theorem 4.7, we get

- If n is even then the log coefficient (5.40) is

$$\frac{Q^{g_\Sigma^\mu}}{2^{n-1} \left(\frac{n}{2} - 1\right)! \left(\frac{n}{2}\right)!}$$

where $Q^{g_\Sigma^\mu}$ is the Q-curvature of g_Σ^μ .

- If n is odd then the log term vanishes, and the solution is smooth.

In fact (5.41) is precisely the problem studied by [21] and [23] (noting the different sign of Laplacian these sources use) and these results are consistent with their findings.

5.6. The formal solutions in the interior scale. Summarising the above, away from the weights $w_0 = \{\frac{j-d}{2} \mid j \in \mathbb{Z}_{\geq 1}\}$, we found two independent formal solutions to the equation $I \cdot Df = 0$ along Σ . These take the form

$$F = f_0 + \sigma f_1 + \cdots \quad \text{and} \quad \sigma^{h_0-1} G = \sigma^{h_0-1} (\bar{f}_0 + \sigma \bar{f}_1 + \cdots),$$

each of which is a density valued section of $\mathcal{E}^\Phi[w_0]$ with $w_0 \notin \{\frac{j-d}{2} \mid j \in \mathbb{Z}_{\geq 1}\}$.

We may eliminate the weight by choosing a scale $\tau \in \Gamma\mathcal{E}[1]$ which extends to the boundary Σ , so equivalently a metric $\bar{g} = \tau^{-2} g \in c$. Then the solutions, in terms of the scale \bar{g} , are $F^{\bar{g}} = \tau^{-w_0} F$ and $G^{\bar{g}} = \tau^{-w_0} G$ in $\mathcal{E}^\Phi = \mathcal{E}^\Phi[0]$. Note that $r := \sigma/\tau$ is the

function r which gives the density σ in the scale τ . Thus working in the metric \bar{g} the solutions take the form

$$F^{\bar{g}} = f_0 + rf_1 + \cdots \quad \text{and} \quad [\sigma^{h_0-1}G]^{\bar{g}} = r^{h_0-1}(\bar{f}_0 + r\bar{f}_1 + \cdots)$$

where now the f_i and \bar{f}_i are of weight 0 (but we have retained the earlier notation for simplicity). Explicit expressions for the unweighted tractor fields f_i and \bar{f}_i follow from Proposition 5.11.

On the other hand, for the purpose of comparison with the existing literature, it is interesting to understand the solutions in terms of the canonical generalised scale σ . Away from Σ , we have $F^{g^o} = \sigma^{-w_0}F$ or in terms of the scale \bar{g} , $F^{g^o} = r^{-w_0}F^{\bar{g}}$, and similarly for \bar{f}^{g^o} which may then be expressed in terms of $G^{\bar{g}}$. Thus, in terms of the canonical scale g^o , a general formal solution takes the form

$$f = r^{n-s}F^{\bar{g}} + r^sG^{\bar{g}}$$

where, recall, $s := w_0 + n$. This equation (specialized to conformal densities rather than generic tractors), alongside the expression (2.10) for the $I \cdot D$ operator in the scale g^o , gives a precise dictionary between our methods and the scattering problem stated in equation 3.2 of [40].

At weights $w_0 \in \{\frac{j-d}{2} \mid j \in \mathbb{Z}_{\geq 1}\}$ where log solutions appear, for the scale $\bar{g} = \tau^{-2}g$ we take the density τ extending to the boundary to be the same as that appearing in the log terms in (5.15). In that choice of scale τ itself is represented by the constant function 1, thus a similar analysis to above shows that in the scale g^o we have solutions of the form

$$(5.42) \quad f = r^{n-s}F_{h_0-2}^{\bar{g}} + r^s \log r \bar{F}^{\bar{g}} + r^{s+1}B^{\bar{g}} + r^sG^{\bar{g}},$$

where now $s \in \{\frac{j+n-1}{2} \mid j \in \mathbb{Z}_{\geq 1}\}$, the field $G^{\bar{g}}$ (the “second solution” part) is of the form as above, and the fields $F_{h_0-2}^{\bar{g}}$, $\bar{F}^{\bar{g}}$ and $B^{\bar{g}}$ take the form

$$F_{h_0-2}^{\bar{g}} = f_0 + rf_1 + \cdots + r^{2s-n-1}f_{2s-n-1}, \quad \bar{F}^{\bar{g}} = \bar{f}_0 + r\bar{f}_1 + \cdots$$

and

$$B^{\bar{g}} = b_0 + rb_1 + \cdots$$

These last three mentioned terms are really part of the single log type solution and the details can be explicitly read off from Theorem 5.18. In the special case $s = \frac{n}{2}$ (so that $h_0 = 1$), the polynomial term $F_{h_0-2}^{\bar{g}}$ is omitted. For comparison with the existing literature, note that the non-log terms $r^{n-s}F_{h_0-2}^{\bar{g}}$, $r^{s+1}B^{\bar{g}}$, and $r^sG^{\bar{g}}$ in (5.42) could be swept together into a single term of the form $r^{n-s}F$, with F of the form $f_0 + rf_1 + \cdots$, so we have

$$f = r^{n-s}F + r^s \log r \bar{F}^{\bar{g}},$$

but this hides how independent solutions combine to give the general solution.

Continuing in this weight range $w_0 \in \{\frac{j-d}{2} \mid j \in \mathbb{Z}_{\geq 1}\}$, for almost Einstein structures (e.g. Poincaré-Einstein, the case considered in [40]), when h_0 is even so $s \in \{\frac{j+n-1}{2} \mid j \in 2\mathbb{Z}_{\geq 1}\}$ there are no log terms in the solution, as explained in Remark 5.19. Indeed, in [19, 40], by differing methods, the authors found a log obstruction only for $2s - n$ even. Also, essentially, the same analysis as above applied to the log density problem of Section 5.5 confirms the form of solution predicted for the equivalent interior problem formulated in [21].

Finally, to complete a circle of ideas, let us write out the operator $I \cdot D$ in the scale τ such that the function r above is a defining function in the sense of [38]. In that case the metric $g^o = \frac{\bar{g}}{r^2}$ where \bar{g} extends to the boundary and has normal form

$$\bar{g} = dr^2 + h_r.$$

Then, an easy application of (2.9) gives the expression for $I \cdot D$ acting on scalar densities of weight w and we find

$$I \cdot D \frac{\bar{g}}{r^2} = -(r\partial_r - 2s + n + 1)\partial_r - H_r(r\partial_r + n - s) - r\Delta_{h_r}.$$

with $H_r = \frac{1}{2}h^{ij}\partial_r h_{ij}$ and h_{ij} is the metric defined by h_r at fixed r . This is the operator employed in the proof of Proposition 4.2 of [40].

6. THE FEFFERMAN–GRAHAM AMBIENT METRIC AND ALMOST EINSTEIN SPACES

Recall from Section 2.1 that a conformal structure (of any signature (p, q)) is equivalent to the ray bundle $\pi : \mathcal{G} \rightarrow M^d$ of conformally related metrics. Let us use ρ to denote the \mathbb{R}_+ action on \mathcal{G} given by $\rho(t)(x, g_x) = (x, t^2 g_x)$. An *ambient manifold* is a smooth $(d+2)$ -manifold \tilde{M} endowed with a free \mathbb{R}_+ -action ρ and an \mathbb{R}_+ -equivariant embedding $i : \mathcal{G} \rightarrow \tilde{M}$. We write $\mathbf{X} \in \Gamma(T\tilde{M})$ for the fundamental field generating the \mathbb{R}_+ -action. That is, for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$, we have $\mathbf{X}f(u) = (d/ds)f(\rho(e^s)u)|_{s=0}$. For an ambient manifold \tilde{M} , an *ambient metric* is a pseudo-Riemannian metric \mathbf{h} of signature $(p+1, q+1)$ on \tilde{M} satisfying the conditions: (i) $\mathcal{L}_{\mathbf{X}}\mathbf{h} = 2\mathbf{h}$, where $\mathcal{L}_{\mathbf{X}}$ denotes the Lie derivative by \mathbf{X} ; (ii) for $u = (x, g_x) \in \mathcal{G}$ and $\xi, \eta \in T_u\mathcal{G}$, we have $\mathbf{h}(i_*\xi, i_*\eta) = g_x(\pi_*\xi, \pi_*\eta)$. In [20] (and see [22]) Fefferman and Graham considered formally the Goursat problem of obtaining $\text{Ric}(\mathbf{h}) = 0$. They proved that for the case of $d = 2$ and $d \geq 3$ odd this may be achieved to all orders, while for $d \geq 4$ even, the problem is obstructed at finite order by a natural 2-tensor conformal invariant (this is the Bach tensor if $d = 4$, and is called the Fefferman–Graham obstruction tensor in higher even dimensions); for d even one may obtain $\text{Ric}(\mathbf{h}) = 0$ up to the addition of terms vanishing to order $d/2 - 1$. (See [22] for the statements concerning uniqueness. For extracting results via tractors we do not need this, as discussed in *e.g.* [10, 31].) We shall henceforth call any (approximately or otherwise) Ricci-flat metric on \tilde{M} a *Fefferman–Graham metric*. In the subsequent discussion of ambient metrics all results can be assumed to hold formally to all orders unless stated otherwise.

In the following discussion we typically use bold symbols or tilded symbols for the objects on \tilde{M} . For example $\tilde{\nabla}$ denotes the Levi-Civita connection on \tilde{M} . Familiarity with the treatment of the Fefferman–Graham metric, as in *e.g.* [10, 32] or [6], will be assumed. In particular, we shall use that suitably homogeneous tensor fields of $\tilde{M}|_{\mathcal{G}}$ correspond to tractor fields. This correspondence is compatible with the Levi-Civita connection in that each weight zero tractor field F on M corresponds to (the restriction to \mathcal{G} of) a homogenous tensor field \mathbf{F} (of the same tensor rank) on \tilde{M} with the property that it is parallel in the vertical direction, that is $\tilde{\nabla}_{\mathbf{X}}\mathbf{F} = \mathbf{X}^A\tilde{\nabla}_A\mathbf{F} = 0$ along \mathcal{G} . The metric \mathbf{h} and its Levi-Civita connection $\tilde{\nabla}$ on \tilde{M} determine a metric and connection on the tractor bundles, and by [10, Theorem 2.5] this agrees with the normal tractor metric and connection. We use abstract indices in an obvious way on \tilde{M} and these are lowered and raised using \mathbf{h}_{AB} and its inverse \mathbf{h}^{AB} .

We shall say \mathbf{F} is homogeneous of *weight* $w_0 \in \mathbb{C}$ if $\tilde{\nabla}_{\mathbf{X}}\mathbf{F} = w_0\mathbf{F}$, and this corresponds to a tractor field F of weight w_0 . We shall always take such fields to be extended off \mathcal{G} smoothly and so that $\tilde{\nabla}_{\mathbf{X}}\mathbf{F} = w_0\mathbf{F}$ on \tilde{M} . Along \mathcal{G} then $\tilde{\nabla}_{\mathbf{X}}$, as applied to tensor fields

of well defined weight, gives an ambient realisation of the weight operator. Note also that if $\tau \in \Gamma\mathcal{E}_+[w_0]$ and $\boldsymbol{\tau}$ is the corresponding ambient function, positive and homogeneous of weight w_0 , then

$$[\boldsymbol{\nabla}_{\mathbf{X}}, \log \boldsymbol{\tau}] = w_0,$$

compatible with our extension of the weight operator to log densities.

In this picture the operator $\mathbf{D}_A = (d + 2w_0 - 2)\boldsymbol{\nabla}_A - \mathbf{X}_A \boldsymbol{\Delta}$ on tensors homogeneous of weight w_0 corresponds to the tractor- D operator as applied to tractors of weight w_0 . Thus we equivalently view this as a restriction of

$$\mathbf{D}_A = \boldsymbol{\nabla}_A(d + 2\boldsymbol{\nabla}_{\mathbf{X}} - 2) + \mathbf{X}_A \boldsymbol{\Delta}.$$

This acts tangentially along the submanifold \mathcal{G} in \tilde{M} , [6] and [32], and is compatible with our treatment of log-densities.

Given a (generalised) scale $\sigma \in \Gamma\mathcal{E}[1]$ on M , we shall write $\tilde{\sigma}$ for the corresponding homogeneous weight 1 function on \mathcal{G} and its homogeneous extension to \tilde{M} . Then with

$$\mathbf{I}_A := \frac{1}{d}\mathbf{D}_A\tilde{\sigma}$$

a restriction of the differential operator

$$(6.1) \quad \mathbf{I} \cdot \mathbf{D} := \boldsymbol{\nabla}_{\mathbf{I}}(d + 2\boldsymbol{\nabla}_{\mathbf{X}} - 2) - \tilde{\sigma} \boldsymbol{\Delta}$$

(on \tilde{M}) lifts the operator $\mathbf{I} \cdot \mathbf{D}$ on M , enabling calculations on \tilde{M} . An important example of such is to compute the commutator

$$\begin{aligned} [\mathbf{I} \cdot \mathbf{D}, \log \tilde{\sigma}] &= 2\boldsymbol{\nabla}_{\mathbf{I}} + \frac{(\boldsymbol{\nabla}_{\mathbf{I}}\tilde{\sigma})}{\tilde{\sigma}}(d + 2\boldsymbol{\nabla}_{\mathbf{X}} - 2) \\ &\quad - \tilde{\sigma}(\boldsymbol{\nabla}_A \frac{(\boldsymbol{\nabla}^A \tilde{\sigma})}{\tilde{\sigma}} + \frac{(\boldsymbol{\nabla}_A \tilde{\sigma})}{\tilde{\sigma}} \boldsymbol{\nabla}^A) \\ &= -\frac{2}{d}(\boldsymbol{\Delta}\tilde{\sigma})\boldsymbol{\nabla}_{\mathbf{X}} + \frac{(\boldsymbol{\nabla}^A \tilde{\sigma})(\boldsymbol{\nabla}_A \tilde{\sigma}) - \frac{1}{d}\tilde{\sigma}(\boldsymbol{\Delta}\tilde{\sigma})}{\tilde{\sigma}}(d + 2\boldsymbol{\nabla}_{\mathbf{X}} - 2) \\ &\quad - (\boldsymbol{\Delta}\tilde{\sigma}) + \frac{(\boldsymbol{\nabla}^A \tilde{\sigma})(\boldsymbol{\nabla}_A \tilde{\sigma})}{\tilde{\sigma}} \\ &= \frac{(\boldsymbol{\nabla}^A \tilde{\sigma})(\boldsymbol{\nabla}_A \tilde{\sigma}) - \frac{2}{d}\tilde{\sigma}(\boldsymbol{\Delta}\tilde{\sigma})}{\tilde{\sigma}}(d + 2\boldsymbol{\nabla}_{\mathbf{X}} - 1). \end{aligned}$$

Here the first line used the identity $[\boldsymbol{\nabla}_{\mathbf{X}}, \log \tilde{\sigma}] = 1$ while the second equality relies on the expression for the ambient scale tractor $\mathbf{I}_A = \boldsymbol{\nabla}_{\mathbf{I}}\tilde{\sigma} - \frac{1}{d}\mathbf{X}_A \boldsymbol{\Delta}\tilde{\sigma}$. From this we recognize that the last the line of the above equals $\frac{\mathbf{I}^2}{\tilde{\sigma}}(d + 2\boldsymbol{\nabla}_{\mathbf{X}} - 1)$, along \mathcal{G} , and have therefore, remembering that $y = -\frac{1}{I^2}\mathbf{I} \cdot \mathbf{D}$, established the following result.

Proposition 6.1. *Suppose that (M, c, σ) is such that I^2 is nowhere vanishing. Then on any section of a weighted tractor bundle*

$$[y, \log x] = -x^{-1}(h - 1).$$

6.1. The almost Einstein setting and Theorem 4.5. Here we consider the case (M^d, c, σ) almost Einstein, so $\mathbf{I}_A = \frac{1}{d}\mathbf{D}_A\sigma$ is parallel. Let us first consider the case of d odd. The homogeneous degree 1 function $\tilde{\sigma}$ on \mathcal{G} , corresponding to σ , can be extended off \mathcal{G} smoothly to a homogeneous degree 1 function on \tilde{M} that also satisfies $\boldsymbol{\Delta}\tilde{\sigma} = 0$ formally to all orders [36]. In this case, which we henceforth assume, \mathbf{I}_A is parallel

on \tilde{M} [28]. (Reference [28] treats the case of Riemannian signature (M, c) , and $I^2 = 1$, there is little difference in moving to general signature and constant $I^2 \neq 0$.)

Let us assume then that the ambient manifold \tilde{M} is equipped with a parallel vector $\mathbf{I}^A = \frac{1}{d} \mathbf{D}^A \tilde{\sigma}$, of non-zero length, and the zero locus $\tilde{\Sigma} = \mathcal{Z}(\tilde{\sigma})$ of $\tilde{\sigma} = \mathbf{X}^A \mathbf{I}_A$ is non-empty. Note that $\nabla \tilde{\sigma}$ is nowhere vanishing on $\tilde{\Sigma}$, which is thus necessarily a hypersurface. By homogeneity, $\tilde{\Sigma} \cap \mathcal{G} = \pi^{-1}(\Sigma)$, where $\Sigma = \mathcal{Z}(\sigma)$.

Let \tilde{f}_0 be a section of a tensor bundle on \tilde{M} which is homogeneous of weight $\frac{k-n}{2}$, where $n = d - 1$. We wish to consider $(\mathbf{I} \cdot \mathbf{D})^k \tilde{f}_0$ along $\tilde{\Sigma}$. By the lift of the calculations from Section 4.1 we conclude that $(\mathbf{I} \cdot \mathbf{D})^k$ acts tangentially on \tilde{f}_0 , along $\tilde{\Sigma} \cap \mathcal{G}$. In fact it is easily verified that $\tilde{\sigma}$, $-\frac{1}{I^2} \mathbf{I} \cdot \mathbf{D}$ and $(d + 2\nabla_{\mathbf{X}})$ generate an $\mathfrak{sl}(2)$ (compatible in the obvious way with \mathfrak{g}) and that $(\mathbf{I} \cdot \mathbf{D})^k$ acts tangentially on \tilde{f}_0 , along $\tilde{\Sigma}$. Thus to simplify calculations we extend smoothly \tilde{f}_0 smoothly off $\tilde{\Sigma}$ by the equation $\mathbf{I}^A \nabla_A \tilde{f}_0 = 0$. It is easily verified that (for any weight w_0) this is consistent with the homogeneity assumption $\nabla_{\mathbf{X}} \tilde{f}_0 = w_0 \tilde{f}_0$.

For \tilde{f}_0 of any weight w_0 , and satisfying $\mathbf{I}^A \nabla_A \tilde{f}_0 = 0$ we have

$$\begin{aligned} (\mathbf{I} \cdot \mathbf{D})^k \tilde{f}_0 &= -(\mathbf{I} \cdot \mathbf{D})^{k-1} (\tilde{\sigma} \Delta \tilde{f}_0) \\ &= -\tilde{\sigma} (\mathbf{I} \cdot \mathbf{D})^{k-1} \tilde{\Delta} \tilde{f}_0 - I^2 (k-1)(d+2w_0-k-2)(\mathbf{I} \cdot \mathbf{D})^{k-2} \Delta \tilde{f}_0. \end{aligned}$$

On the other hand using that \mathbf{I} is parallel, we have that $[\mathbf{I}^A \nabla_A, \Delta] = 0$, on tensor fields over \tilde{M} . These observations yield the following useful identity.

Lemma 6.2.

$$\begin{aligned} (\mathbf{I} \cdot \mathbf{D})^{k-2j} \Delta^j \tilde{f}_0 &= -\tilde{\sigma} (\mathbf{I} \cdot \mathbf{D})^{k-2j-1} \Delta^{j+1} \tilde{f}_0 \\ &\quad - I^2 (k-2j-1)(d+2w_0-k-2j-2)(\mathbf{I} \cdot \mathbf{D})^{k-2j-2} \Delta^{j+1} \tilde{f}_0, \end{aligned}$$

where w_0 is the homogeneity weight of \tilde{f}_0 .

From this we obtain the following result.

Proposition 6.3. *For \tilde{f}_0 any tensor field, of homogeneity weight $w_0 = \frac{k-d+1}{2}$, along $\tilde{\Sigma}$ we have*

$$(\mathbf{I} \cdot \mathbf{D})^k \tilde{f}_0 = \begin{cases} [(k-1)!!]^2 (I^2 \Delta_{\tilde{\Sigma}})^{\frac{k}{2}} \tilde{f}_0 + O(\tilde{\sigma}), & k \text{ even}, \\ O(\tilde{\sigma}), & k \text{ odd}. \end{cases},$$

where $\Delta_{\tilde{\Sigma}}$ is the induced intrinsic Laplacian of $\tilde{\Sigma}$ coupled to the Levi-Civita connection of \tilde{M} .

Proof. Since $(\mathbf{I} \cdot \mathbf{D})^k$ acts tangentially there is no loss of generality in assuming that $\mathbf{I}^A \nabla_A \tilde{f}_0 = 0$. Then from the Lemma and an obvious induction we obtain

$$(\mathbf{I} \cdot \mathbf{D})^k \tilde{f}_0 = \begin{cases} [(k-1)!!]^2 (I^2 \Delta)^{\frac{k}{2}} \tilde{f}_0 + O(\tilde{\sigma}), & k \text{ even}, \\ O(\tilde{\sigma}), & k \text{ odd}. \end{cases}$$

But using that along $\tilde{\Sigma}$ we have $\Delta = \Delta_{\tilde{\Sigma}} + \frac{1}{I^2} \mathbf{I}^A \mathbf{I}^B \nabla_A \nabla_B$ and using again that $[\mathbf{I}^A \nabla_A, \Delta] = 0$ the result follows immediately. \square

We see that for k odd, on an almost Einstein manifold P_k is the zero operator on Σ , as claimed in Proposition 4.4. The statement there concerning the leading order term of the operator is also an easy consequence of the calculation above.

Now for simplicity let us assume $I^2 = 1$. As established in [28, Theorem 6.3] the induced Ricci metric $\mathbf{h}_{\tilde{\Sigma}}$ on $\tilde{\Sigma}$ is Ricci flat and $(\tilde{\Sigma}, \mathbf{h}_{\tilde{\Sigma}})$ gives a Fefferman–Graham ambient metric to all orders for the conformal structure (Σ, c_{Σ}) .

Now consider the case where f_0 is a density in $\mathcal{E}[\frac{k-n}{2}]$ and \tilde{f}_0 the corresponding homogeneous function on \tilde{M} . On the Fefferman–Graham space $(\tilde{\Sigma}, \mathbf{h}_{\tilde{\Sigma}})$ the powers $\Delta_{\tilde{\Sigma}}^{\frac{k}{2}}$, for even k , are tangential along $\mathcal{G} \cap \tilde{\Sigma}$; for $k \leq n$ these give the GJMS operators, by definition [36]. It follows at once that for f_0 a density in $\mathcal{E}[\frac{k-n}{2}]$ we have

$$\left(-\frac{1}{I^2} \mathbf{I} \cdot \mathbf{D}\right)^k f_0 = (-1)^k [(k-1)!!]^2 \mathcal{P}_k \tilde{f}_0, \quad k \in \{2, 4, \dots, n/2\}$$

where \mathcal{P}_k is the order k GJMS operator. This establishes Theorem 4.5 for $d \geq 3$ odd.

For $d \geq 4$ even all the above goes through, except that the usual construction as in [20], for the Fefferman–Graham ambient metric is obstructed at finite order, as stated earlier. So also is the “harmonic” extension of $\tilde{\sigma}$ off \mathcal{G} , we may obtain for example that $0 = \Delta \tilde{\sigma} = \Delta^2 \tilde{\sigma} = \dots = \Delta^{d/2} \tilde{\sigma}$ along \mathcal{G} , but at the next order the problem is potentially obstructed (and $\Delta^{d/2+1}$ depends on c beyond the order of the obstruction). However by a straightforward counting of derivatives it is verified that we recover the usual GJMS operators to at least the order claimed in Theorem 4.5.

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